

# INDEX THEOREMS FOR COUPLES OF HOLOMORPHIC SELF-MAPS

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**ABSTRACT.** Let  $M$  be a  $n$ -dimensional complex manifold and  $f, g : M \rightarrow M$  two distinct holomorphic self-maps. Suppose that  $f$  and  $g$  coincide on a globally irreducible compact hypersurface  $S \subset M$ . We show that if one of the two maps is a local biholomorphism around  $S' = S - \text{Sing}(S)$  and, if needed,  $S'$  sits into  $M$  in a particular nice way, then it is possible to define a 1-dimensional holomorphic (possibly singular) foliation on  $S'$  and partial holomorphic connections on certain holomorphic vector bundles on  $S'$ . As a consequence, we are able to localize suitable characteristic classes and thus to get index theorems.

**Key words and phrases:** Index theorem; Holomorphic foliation; Holomorphic connection; Couple of holomorphic self-maps; Residue.

## INTRODUCTION

This paper is deeply inspired to the various works about index theorems for holomorphic self-maps and holomorphic foliations. Our goal here is to prove index theorems for couples  $(f, g)$  of holomorphic self-maps coinciding on a positive dimensional set.

A first example of index theorem for holomorphic self-maps is the classical holomorphic Lefschetz fixed-point formula (see for example [13, Ch.3, Sec.4]) which regards maps  $f : M \rightarrow M$  having isolated fixed points, with  $M$  a compact complex manifold. Anyway, we are mostly inspired by index theorems concerning self-maps having a positive dimensional fixed-points set, like for example the one in [1]. In his paper Abate obtained a complete generalization to two complex variables of the classical Leau-Fatou flower theorem for maps tangent to the identity and a key ingredient in the proof was an index theorem for holomorphic self-maps on complex surfaces  $M$  fixing pointwise a non-singular compact complex curve  $S \subset M$ . This theorem was inspired by the Camacho-Sad index theorem for invariant leaves of possibly singular holomorphic foliations on complex surfaces (see [12, Appendix]), later generalized to possibly singular leaves first (see [19]) and then to arbitrary dimension of the ambient complex manifold, the foliation and the leaves (see [15], [16], [17] or [20] for a complete treatment). Similarly, a first generalization of Abate index theorem was made in [10] (see also [8]) where the authors assume  $S$  to be possibly singular. However a large generalization to any dimension of  $M$  and codimension of the possibly singular  $S \subset M$  was

made by Abate-Bracci-Tovena in [3], where they proved even other index theorems.

In this paper we do a step further because we replace the single holomorphic self-map  $f : M \rightarrow M$  pointwise fixing an analytic sub-variety  $S \subset M$  with a couple  $(f, g)$  of distinct holomorphic self-maps coinciding on  $S$  (which, for simplicity, we assume of codimension 1). Clearly if one considers the couple  $(f, \text{Id}_M)$  then falls back in the [3] case. Briefly, we show that assuming some hypotheses on the couple  $(f, g)$  and possibly on the hypersurface  $S$  one can define a 1-dimensional holomorphic foliation on  $S' = S - \text{Sing}(S)$  and certain partial holomorphic connections (outside a ‘singular set’) on suitable holomorphic vector bundles on  $S'$ . As a consequence one can use the Lehmann-Suwa machinery (see [20] or [11] for a systematic exposition) in order to gain index theorems.

Our index theorems generalize the ones in [3] and may be seen as versions for couples of holomorphic self-maps of some main index theorems of foliation theory. To be precise, we get new versions of the Baum-Bott index theorem (see [6, Th.1.] or [20, Th.III.7.6.]), of the above cited Camacho-Sad index theorem and of the Lehmann-Suwa (or variation) theorem (see [17] or [20, Th.IV.5.6.]). We point out that for the last two cited index theorems of foliation theory one needs to have a foliation defined around  $S$  leaving it invariant, while we are able to define foliations on  $S$  only. However, the foliations we define extend naturally to a suitable infinitesimal neighborhood of the sub-variety, and this allows to localize certain characteristic classes producing our index theorems (see also [4] for a general explanation).

The plan of the paper and our main theorems are the following. In Section 1 we introduce the ‘order of coincidence of  $(f, g)$  along  $S$ ’, which is a positive integer denoted by  $\nu_{f,g}$ , and then we define the ‘canonical section associated to  $(f, g)$ ’

$$\mathcal{D}_{f,g} : N_{S'}^{\otimes \nu_{f,g}} \longrightarrow TM|_{S'}$$

on the regular part  $S'$  of  $S$ . In order to define the canonical section we need to assume that one of the two maps is a local biholomorphism on a whole neighborhood of  $S'$  and we will make this assumption for the rest of the paper. In the following Section 2 we introduce two hypotheses under which the canonical section associated to  $(f, g)$  induces a 1-dimensional holomorphic foliation on  $S'$  (or more than one in some cases), denoted here by

$$\mathcal{D} : N_{S'}^{\otimes \nu_{f,g}} \longrightarrow TS'.$$

One hypothesis regards the couple  $(f, g)$ , which is said ‘tangential along  $S$ ’ when it is satisfied. The other concerns the way  $S'$  sits into  $M$  and in this case we say that ‘ $S'$  splits into  $M$ ’. Subsequently, we prove in Section 3 that when  $S$  is non-singular and one of the two cited hypotheses is verified one can define in a canonical way a partial holomorphic connection (in the sense of Bott [7])  $\delta_{\mathcal{D}}^{bb}$  on the normal bundle  $N_{\mathcal{D}}$  of the foliation. As a consequence we get an index theorem which we state here in a simplified version.

**Theorem 0.1.** *Let  $M$  be a  $n$ -dimensional complex manifold,  $S \subset M$  a non-singular compact connected complex hypersurface in  $M$  and  $(f, g)$  a couple of holomorphic self-maps on  $M$  such that  $f|_S = g|_S$  and  $g$  is a local biholomorphism on a neighborhood of  $S$ . Assume that*

(i)  *$(f, g)$  is tangential along  $S$*

*or that*

(ii)  *$S$  splits into  $M$*

*and let  $\mathcal{D}$  be the foliation on  $S$  induced by the canonical section  $\mathcal{D}_{f,g}$ . Assume  $\mathcal{D} \neq 0$  and let  $\text{Sing}(\mathcal{D}) = \sqcup_\lambda \Sigma_\lambda$  be the decomposition in connected components of the singular set of the foliation.*

*Then for any symmetric homogeneous polynomial  $\varphi \in \mathbb{C}[z_1, \dots, z_{n-2}]$  of degree  $n-1$  there exist complex numbers  $\text{Res}_\varphi(\mathcal{D}; TS - N_S^{\otimes \nu}; \Sigma_\lambda)$  such that*

$$\sum_{\lambda} \text{Res}_\varphi(\mathcal{D}; TS - N_S^{\otimes \nu}; \Sigma_\lambda) = \int_S \varphi(TS - N_S^{\otimes \nu}).$$

We conclude by computing the residues appearing in Theorem 0.1 at isolated singular points of the foliation. In Section 4 we show that if  $(f, g)$  is tangential along  $S$  then one can locally extend the foliation  $\mathcal{D}$  on  $S'$  in a ‘canonical way’ and control in a sense the differences among canonical local extensions. We see moreover that if  $(f, g)$  is not tangential along  $S$  one can still do this but assuming a hypothesis on  $S'$  stronger than the splitting property. When this condition is satisfied we say that ‘ $S'$  is comfortably embedded into  $M$ ’. Finally, in the last two sections we reap the benefits of Section 4. Indeed in Section 5 we show that these canonical local extensions of  $\mathcal{D}$  are good enough to define (outside a ‘singular set’) a partial holomorphic connection  $\delta_{\mathcal{D}}^{\text{cs}}$  on the normal bundle  $N_{S'}$  of  $S'$  in  $M$ . The result is the following theorem.

**Theorem 0.2.** *Let  $M$  be a  $n$ -dimensional complex manifold,  $S \subset M$  a globally irreducible compact complex hypersurface and  $(f, g)$  a couple of holomorphic self-maps on  $M$  such that  $f|_S = g|_S$  and  $g$  is a local biholomorphism on a neighborhood of  $S'$ . Assume that*

(i)  *$(f, g)$  is tangential along  $S$*

*or that*

(ii)  *$S'$  is comfortably embedded into  $M$*

*and let  $\mathcal{D}$  be the foliation on  $S'$  induced by the canonical section  $\mathcal{D}_{f,g}$ . Assume  $\mathcal{D} \neq 0$  and let  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D}) = \sqcup_\lambda \Sigma_\lambda$  be the decomposition in connected components of the singular set  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D})$ .*

*Then there exist complex numbers  $\text{Res}(\mathcal{D}; S; \Sigma_\lambda)$  such that*

$$\sum_{\lambda} \text{Res}(\mathcal{D}; S; \Sigma_\lambda) = \int_S c_1^{n-1}([S]),$$

*where  $c_1([S])$  denotes the first Chern class of the line bundle  $[S]$  on  $M$  canonically associated to  $S$ .*

Again, we conclude by computing the residues appearing in Theorem 0.2 at isolated singular points. Similarly, in Section 6 we show that if  $(f, g)$  is tangential along  $S$  and  $\nu_{f,g} > 1$  then the canonical local extensions of  $\mathcal{D}$  are good enough to define a partial holomorphic connection  $\delta_{\mathcal{D}}^{ls}$  on the normal bundle  $N_{\mathcal{D}}^M$  of the foliation respect to the ambient tangent bundle  $TM$  (restricted to  $S'$ ). It follows the last index theorem.

**Theorem 0.3.** *Let  $M$  be a  $n$ -dimensional complex manifold,  $S \subset M$  a globally irreducible compact complex hypersurface and  $(f, g)$  a couple of holomorphic self-maps on  $M$  such that  $f|_S = g|_S$  and  $g$  is a local biholomorphism on a neighborhood of  $S'$ . Suppose  $(f, g)$  tangential along  $S$  and  $\nu_{f,g} > 1$  and let  $\mathcal{D}$  be the foliation on  $S'$  induced by the canonical section. Assume  $\mathcal{D} \neq 0$  and let  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D}) = \sqcup_{\lambda} \Sigma_{\lambda}$  be the decomposition in connected components of the singular set  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D})$ .*

*Then for any symmetric homogeneous polynomial  $\varphi \in \mathbb{C}[z_1, \dots, z_{n-1}]$  of degree  $n - 1$  there exist complex numbers  $\text{Res}_{\varphi}(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; \Sigma_{\lambda})$  such that*

$$\sum_{\lambda} \text{Res}(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; \Sigma_{\lambda}) = \int_S \varphi(TM - [S]^{\otimes \nu}).$$

As for the other index theorems we end by computing the residues appearing in Theorem 0.3 at isolated singular points.

## 1. THE ORDER OF COINCIDENCE AND THE CANONICAL SECTION

Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a (possibly singular) globally irreducible complex hypersurface. From now on we will denote by  $\mathcal{O}_M$  the sheaf of germs of holomorphic functions on  $M$  and by  $\mathcal{I}_S$  the sub-sheaf of germs of functions vanishing on  $S$ , which is a coherent  $\mathcal{O}_M$ -module. For the sake of simplicity we shall use the same symbol to denote both a germ at some point and any representative defined in a neighborhood of the point. Recall that the sheaf of germs of holomorphic functions on  $S$  is by definition  $\mathcal{O}_S = \mathcal{O}_M / \mathcal{I}_S$ . We will denote by  $TM$  the holomorphic tangent bundle of  $M$  and, in case  $S$  is non-singular, by  $TS$  the holomorphic tangent bundle of  $S$  and by  $N_S$  its holomorphic normal bundle in  $M$ , which is a line bundle. The corresponding sheaves of germs of holomorphic sections will be denoted respectively by  $\mathcal{T}_M$ ,  $\mathcal{T}_S$  and  $\mathcal{N}_S$ . Lastly, we will denote by  $\text{End}_S^2(M)$  the set of couples  $(f, g)$  of distinct (germs about  $S$  of) holomorphic self-maps of  $M$  *coinciding on  $S$* . In other words, if  $(f, g) \in \text{End}_S^2(M)$  then  $f, g : M \rightarrow M$  are holomorphic,  $f \neq g$ , and  $f|_S \equiv g|_S$ .

The goal of this first section is to introduce the concept of ‘order of coincidence’ of a couple  $(f, g)$  and to define the ‘canonical section’ associated to a couple with a certain property, which is a holomorphic section of a suitable holomorphic vector bundle on  $S' = S - \text{Sing}(S)$ .

So let  $(f, g) \in \text{End}_S^2(M)$  be a given couple and fix a point  $p \in S$ . Observe that for every germ  $h \in \mathcal{O}_{M, f(p)}$  we have the well-defined germ  $h \circ f - h \circ g \in \mathcal{I}_{S, p} \subset \mathcal{O}_{M, p}$ , so we can give the following definition.

**Definition 1.1.** For any  $h \in \mathcal{O}_{M, f(p)}$ , we define  $\nu_{f, g}^p(h)$  to be the constant

$$\nu_{f, g}^p(h) = \max\{\nu \in \mathbb{N} \text{ s.t. } h \circ f - h \circ g \in \mathcal{I}_{S, p}^\nu\}.$$

The order of coincidence of  $(f, g)$  at  $p$  along  $S$  is then

$$\nu_{f, g}^p = \min\{\nu_{f, g}^p(h), \text{ for } h \in \mathcal{O}_{M, f(p)}\}.$$

If  $(w^1, \dots, w^n)$  is any local coordinates system at  $f(p) = g(p)$  then for every  $h \in \mathcal{O}_{M, f(p)}$  we have the fundamental relations of germs at  $p$

$$\begin{aligned} h \circ f - h \circ g &= \sum_{j=1}^n (f^j - g^j) \frac{\partial h}{\partial w^j} \circ g \left( \text{mod } \mathcal{I}_{S, p}^{2\nu_{f, g}^p} \right) = \\ &= \sum_{j=1}^n (f^j - g^j) \frac{\partial h}{\partial w^j} \circ f \left( \text{mod } \mathcal{I}_{S, p}^{2\nu_{f, g}^p} \right) \end{aligned} \quad (1.1)$$

where  $f^j = w^j \circ f$  and  $g^j = w^j \circ g$ . In fact, by Definition 1.1

$$\begin{aligned} f^j - g^j &= s^j \in \mathcal{I}_{S, p}^{\nu_{f, g}^p}, & j &= 1, \dots, n \\ \frac{\partial h}{\partial w^j} \circ f - \frac{\partial h}{\partial w^j} \circ g &\in \mathcal{I}_{S, p}^{\nu_{f, g}^p}, & j &= 1, \dots, n, \end{aligned}$$

then

$$\begin{aligned} h \circ f - h \circ g &= h(g^1 + s^1, \dots, g^n + s^n) - h(g^1, \dots, g^n) = \\ &= \sum_{j=1}^n s^j \frac{\partial h}{\partial w^j} \Big|_{(g^1, \dots, g^n)} + \frac{1}{2} \sum_{j, k=1}^n s^j s^k \frac{\partial^2 h}{\partial w^j \partial w^k} \Big|_{(g^1, \dots, g^n)} + \dots = \\ &= \sum_{j=1}^n (f^j - g^j) \frac{\partial h}{\partial w^j} \circ g \left( \text{mod } \mathcal{I}_{S, p}^{2\nu_{f, g}^p} \right) = \\ &= \sum_{j=1}^n (f^j - g^j) \frac{\partial h}{\partial w^j} \circ f \left( \text{mod } \mathcal{I}_{S, p}^{2\nu_{f, g}^p} \right), \end{aligned}$$

where the second row is given by the Taylor expansion of  $h$  at  $f(p) = g(p)$ .

**Lemma 1.2.** Let  $p \in S$ . Then

i) if  $(w^1, \dots, w^n)$  is any set of local coordinates at  $f(p)$  then

$$\nu_{f, g}^p = \min\{\nu_{f, g}^p(w^1), \dots, \nu_{f, g}^p(w^n)\}.$$

ii) for any  $h \in \mathcal{O}_{M, f(p)}$  the function

$$q \rightarrow \nu_{f, g}^q(h)$$

is constant in a neighborhood of  $p$  in  $S$ .

iii) the function

$$p \rightarrow \nu_{f,g}^p$$

is constant on  $S$ .

*Proof.* i) By Definition 1.1 it follows that  $\nu_{f,g}^p \leq \min \left\{ \nu_{f,g}^p(w^1), \dots, \nu_{f,g}^p(w^n) \right\}$  and by (1.1) it follows the reverse inequality.

ii) Let  $h \in \mathcal{O}_{M,f(p)}$  and  $\{\gamma^1, \dots, \gamma^t\}$  be a set of generators of  $\mathcal{I}_{S,p}$ , then

$$h \circ f - h \circ g = \sum_{|I|=\nu_{f,g}^p(h)} \gamma^I c_I,$$

where  $I = (i_1, \dots, i_t) \in \mathbb{N}^t$  is a multi-index,  $|I| = i_1 + \dots + i_t$ ,  $\gamma^I = (\gamma^1)^{i_1} \dots (\gamma^t)^{i_t}$  and  $c_I \in \mathcal{O}_{M,p}$ . This equality clearly holds in a neighborhood of  $p$  by definition of germ. Moreover, since  $\mathcal{I}_S$  is a coherent sheaf the  $\gamma$ 's are generators of  $\mathcal{I}_{S,q}$  for points  $q$  near enough to  $p$ . Finally, observe that by definition of  $\nu_{f,g}^p(h)$  there is an index  $I_0$  such that  $c_{I_0} \notin \mathcal{I}_{S,p}$ , then  $c_{I_0} \notin \mathcal{I}_{S,q}$  for all  $q \in S$  close enough to  $p$ . All these facts make the assertion follows easily.

iii) By i) and ii) the function  $p \rightarrow \nu_{f,g}^p$  is locally constant on  $S$ . Since  $S$  is connected then it is constant.  $\square$

As a consequence of iii) the following definition makes sense.

**Definition 1.3.** The *order of coincidence of  $(f, g)$  along  $S$*  is the constant

$$\nu_{f,g} = \nu_{f,g}^p$$

for any point  $p \in S$ .

Now assume  $S$  non-singular in the following and let  $(f, g) \in \text{End}_S^2(M)$  be a couple in which one of the two maps, say  $g$ , is a local biholomorphism if restricted to an open neighborhood of  $S$  in  $M$ . This means that there exists an open neighborhood  $W \subset M$  containing  $S$  such that  $g|_W : W \rightarrow M$  is a local biholomorphism, or equivalently that  $dg|_p : T_p M \rightarrow T_{g(p)} M$  is an isomorphism of complex vector spaces for every  $p \in S$ . As  $f|_S \equiv g|_S$  there is a well-defined morphism of holomorphic vector bundles

$$df - dg : N_S \rightarrow (f^*TM)|_S, \quad (1.2)$$

which on the fibers is given by  $[v] \rightarrow [df|_p(v) - dg|_p(v)]$ , for any  $[v] \in T_p M$  and  $p \in S$ . Observe that while in general  $f^*TM \neq g^*TM$  obviously by the assumptions  $(f^*TM)|_S = (g^*TM)|_S$ . By the property of  $g$  we also have the isomorphism of holomorphic vector bundles

$$dg|_S : TM|_S \rightarrow (f^*TM)|_S$$

thus composing its inverse with (1.2) we obtain the morphism

$$dg|_S^{-1} \circ (df - dg) : N_S \rightarrow TM|_S, \quad (1.3)$$

which can also be seen as a holomorphic section of  $N_S^* \otimes TM|_S$ . We want to express (1.3) in a local frame but first we introduce some general terminology about local charts on  $M$ .

**Definition 1.4.** Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a complex sub-manifold of any codimension  $k$  ( $0 < k < n$ ). We say that a local holomorphic chart  $(U, z) = (U, z^1, \dots, z^n)$  is *adapted to  $S$*  if  $U \cap S = \emptyset$  or if  $U \cap S = \{z^1 = \dots = z^k = 0\}$ . Equivalently, we can say that the local coordinates are adapted to  $S$ . If moreover the coordinates are centered at  $p \in U \cap S$  we say that the chart is (or the coordinates are) *adapted to  $S$  at  $p$* . We call an atlas  $\mathfrak{U}$  of  $M$  of local adapted charts an *atlas adapted to  $S$* .

Let  $W \subset M$  be an open neighborhood of  $S$  and  $\varphi : W \rightarrow M$  a local biholomorphism. If  $(U, z)$  is a local chart such that  $\varphi|_U$  is a biholomorphism onto its image we can consider the coordinates  $w = (w^1 = z^1 \circ \varphi^{-1}, \dots, w^n = z^n \circ \varphi^{-1})$  on  $g(U)$ . If  $(U, z)$  is also adapted to  $S$  (at  $p$ ) we say that the coordinates  $z$  are (or the chart  $(U, z)$  is) *adapted to  $(\varphi, S)$  (at  $p$ )* and the coordinates  $w = z \circ \varphi^{-1}$  are called *special*. We call an atlas  $\mathfrak{U}$  of  $M$  of local charts adapted to  $(\varphi, S)$  an *atlas adapted to  $(\varphi, S)$* .

Observe that if  $S \subset M$  is a hypersurface and  $(U, z)$  is a local chart adapted to it such that  $U \cap S \neq \emptyset$  then

$$\left\{ \frac{\partial}{\partial z^2} \Big|_S, \dots, \frac{\partial}{\partial z^n} \Big|_S \right\}$$

is a local holomorphic frame for  $TS$ , while if  $\pi : TM|_S \rightarrow N_S$  is the obvious projection then

$$\partial z^1 = \pi \left( \frac{\partial}{\partial z^1} \Big|_S \right)$$

is a local holomorphic generator for  $N_S$  and we denote with  $\partial^* z^1$  its dual (which is a local holomorphic generator for  $N_S^*$ ).

In order to write (1.3) locally we calculate (1.2) first. So let  $(U, z)$  be a local chart adapted to  $S$  at a point  $p \in S$  and  $(V, w)$  be *any* local chart at  $f(p)$ . Then

$$\left\{ \partial^* z^1 \otimes f^* \frac{\partial}{\partial w^1} \Big|_S, \dots, \partial^* z^1 \otimes f^* \frac{\partial}{\partial w^n} \Big|_S \right\}$$

is a local holomorphic frame for  $N_S^* \otimes (f^* TM)|_S$  on the neighborhood  $U \cap f^{-1}(V) \cap S$  of  $p$ , while

$$\left\{ \partial^* z^1 \otimes \frac{\partial}{\partial z^1} \Big|_S, \dots, \partial^* z^1 \otimes \frac{\partial}{\partial z^n} \Big|_S \right\}$$

is a local holomorphic frame for  $N_S^* \otimes TM|_S$  on  $U \cap S$ . Observe that even if in general  $f^* \frac{\partial}{\partial w^j} \neq g^* \frac{\partial}{\partial w^j}$ , their restrictions to  $S$  are equal by the assumptions. If we denote  $f^j = w^j \circ f$  and  $g^j = w^j \circ g$  then clearly  $f^j - g^j \in \mathcal{I}_{S,p}$  for every  $j$  by the hypothesis. Since the coordinates  $z$  are adapted to  $S$  there

exist germs  $h^j \in \mathcal{O}_{M,p}$  such that  $f^j - g^j = h^j z^1$ , for  $j = 1, \dots, n$ . A trivial calculation shows that (1.2) is locally given by

$$\sum_{j=1}^n \frac{\partial(f^j - g^j)}{\partial z^1} \Big|_S \partial^* z^1 \otimes f^* \frac{\partial}{\partial w^j} \Big|_S = \sum_{j=1}^n h^j|_S \partial^* z^1 \otimes f^* \frac{\partial}{\partial w^j} \Big|_S. \quad (1.4)$$

If  $(U, z)$  is adapted to  $(g, S)$  and we take the associated special coordinates  $w$  on  $V = g(U)$  we can easily compute (1.3). In fact, with this choice of coordinates

$$f^* \frac{\partial}{\partial w^j} \Big|_q = g^* \frac{\partial}{\partial w^j} \Big|_q = \frac{\partial}{\partial w^j} \Big|_{g(q)} = dg|_q \left( \frac{\partial}{\partial z^j} \Big|_q \right)$$

for every  $q \in U \cap S$  and  $j = 1, \dots, n$ , so by (1.4) it follows that (1.3) is locally

$$\sum_{j=1}^n h^j|_S \partial^* z^1 \otimes \frac{\partial}{\partial z^j} \Big|_S. \quad (1.5)$$

By Lemma 1.2 and (1.5) it follows that the morphism (1.3) vanishes identically on  $S$  if and only if  $\nu_{f,g} > 1$ , then it would not be a good ‘canonical section’. The idea is to take local higher order derivatives (respect to  $z^1$ ) of “ $f - g$ ”, not vanishing on  $S$ . To be more precise let  $p \in S$  be any point,  $(U, z)$  a local chart adapted to  $(g, S)$  at  $p$  and take on  $g(U)$  the special coordinates  $(w^1, \dots, w^n)$ . Since the coordinates  $z$  are adapted to  $S$  there exist suitable germs  $h^j \in \mathcal{O}_{M,p}$  such that

$$f^j - g^j = w^j \circ f - w^j \circ g = h^j (z^1)^{\nu_{f,g}} \quad (1.6)$$

for  $j = 1, \dots, n$ . Let define

$$\mathfrak{D}_{f,g} := \sum_{j=1}^n h^j (dz^1)^{\nu_{f,g}} \otimes \frac{\partial}{\partial z^j}, \quad (1.7)$$

where  $(dz^1)^{\nu_{f,g}} = (dz^1)^{\otimes \nu_{f,g}}$ , which is a local holomorphic section of  $(TM^{\otimes \nu_{f,g}})^* \otimes TM$  on  $U$ . Observe that it does not vanish identically on  $S$  by Lemma 1.2.

**Remark 1.5.** A priori the germs  $h^j$  do not have representatives defined on  $U$  but we can assume it (possibly shrinking  $U$ ). From now on we will assume the  $h^j$  to be defined on the whole  $U$ .

We have the following fundamental proposition.

**Proposition 1.6.** *Let  $(U, z)$ ,  $(\hat{U}, \hat{z})$  be two local chart adapted to  $(g, S)$  at  $p$  and  $\mathfrak{D}_{f,g}$ ,  $\hat{\mathfrak{D}}_{f,g}$  the corresponding local section of  $(TM^{\otimes \nu_{f,g}})^* \otimes TM$  defined about  $p$  as in (1.7). Then*

$$\hat{\mathfrak{D}}_{f,g} = \mathfrak{D}_{f,g} \quad (\text{mod } \mathcal{I}_S)$$

where they overlap.



*Proof.* In the following set  $\nu = \nu_{f,g}$  for ease the notation. Since  $z^1$  and  $\hat{z}^1$  are both generators of  $\mathcal{I}_{S,p}$  it follows that

$$\hat{z}^1 = az^1 \quad \text{and then} \quad (d\hat{z}^1)^\nu = a^\nu (dz^1)^\nu \pmod{\mathcal{I}_S} \quad (1.8)$$

for some germ  $a \in \mathcal{O}_{M,p}^*$ . Using (1.1), (1.6) and (1.8) we have that

$$\hat{h}^j a^\nu (z^1)^\nu = \sum_{k=1}^n h^k (z^1)^\nu \left( \frac{\partial \hat{w}^j}{\partial w^k} \circ g \right) \pmod{\mathcal{I}_S^{2\nu}},$$

hence

$$\hat{h}^j a^\nu = \sum_{k=1}^n h^k \frac{\partial \hat{z}^j}{\partial z^k} \pmod{\mathcal{I}_S^\nu}, \quad j = 1, \dots, n, \quad (1.9)$$

since one can easily check that  $\frac{\partial \hat{w}^j}{\partial w^k} \circ g = \frac{\partial \hat{z}^j}{\partial z^k}$ . In particular for  $j = 1$

$$\hat{h}^1 a^\nu = h^1 a \pmod{\mathcal{I}_S}. \quad (1.10)$$

An easy computation shows that

$$\frac{\partial}{\partial \hat{z}^1} = \frac{1}{a} \frac{\partial}{\partial z^1} + \sum_{k=2}^n \frac{\partial z^k}{\partial \hat{z}^1} \frac{\partial}{\partial z^k} \pmod{\mathcal{I}_S}, \quad (1.11)$$

$$\frac{\partial}{\partial \hat{z}^j} = \sum_{k=2}^n \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial}{\partial z^k} \pmod{\mathcal{I}_S}, \quad j = 2, \dots, n. \quad (1.12)$$

Then using (1.8), (1.9), (1.10), (1.11) and (1.12) it follows that

$$\begin{aligned} \hat{\mathfrak{D}}_{f,g} &= \sum_{j=1}^n \hat{h}^j (dz^1)^\nu \otimes \frac{\partial}{\partial \hat{z}^j} = \sum_{j=1}^n \hat{h}^j a^\nu (dz^1)^\nu \otimes \frac{\partial}{\partial \hat{z}^j} \pmod{\mathcal{I}_S} = \\ &= ah^1 (dz^1)^\nu \otimes \frac{\partial}{\partial \hat{z}^1} + \sum_{j=2}^n \sum_{r=1}^n h^r \frac{\partial \hat{z}^j}{\partial z^r} (dz^1)^\nu \otimes \frac{\partial}{\partial \hat{z}^j} \pmod{\mathcal{I}_S} = \\ &= h^1 (dz^1)^\nu \otimes \frac{\partial}{\partial z^1} + ah^1 \sum_{k=2}^n \frac{\partial z^k}{\partial \hat{z}^1} (dz^1)^\nu \otimes \frac{\partial}{\partial z^k} + \\ &+ \sum_{j=2}^n \sum_{r=1}^n \sum_{k=2}^n h^r \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial \hat{z}^j}{\partial z^r} (dz^1)^\nu \otimes \frac{\partial}{\partial z^k} \pmod{\mathcal{I}_S} = \\ &= h^1 (dz^1)^\nu \otimes \frac{\partial}{\partial z^1} + h^1 \sum_{k=2}^n \left[ a \frac{\partial z^k}{\partial \hat{z}^1} + \sum_{j=2}^n \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial \hat{z}^j}{\partial z^1} \right] (dz^1)^\nu \otimes \frac{\partial}{\partial z^k} + \\ &+ \sum_{j=2}^n \sum_{r=2}^n \sum_{k=2}^n h^r \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial \hat{z}^j}{\partial z^r} (dz^1)^\nu \otimes \frac{\partial}{\partial z^k} \pmod{\mathcal{I}_S} \end{aligned} \quad (1.13)$$

To conclude observe that for  $k = 2, \dots, n$

$$0 = \frac{\partial z^k}{\partial z^1} = \sum_{j=1}^n \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial \hat{z}^j}{\partial z^1} = a \frac{\partial z^k}{\partial \hat{z}^1} + \sum_{j=2}^n \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial \hat{z}^j}{\partial z^1} \pmod{\mathcal{I}_S}$$

and that

$$\delta_{kr} = \frac{\partial z^k}{\partial z^r} = \sum_{j=1}^n \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial \hat{z}^j}{\partial z^r} = \sum_{j=2}^n \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial \hat{z}^j}{\partial z^r} \pmod{\mathcal{I}_S},$$

for  $r = 2, \dots, n$ . Putting these relations into (1.13) we have done.  $\square$

Thanks to Proposition 1.6 we are now able to define the ‘canonical section’.

**Definition 1.7.** Let  $M$  be a  $n$ -dimensional complex manifold,  $S \subset M$  a non-singular connected complex hypersurface and  $(f, g) \in \text{End}_S^2(M)$  a couple with  $g$  a local biholomorphism if restricted to an open neighborhood of  $S$ .

The *canonical section associated to*  $(f, g)$  is the global holomorphic section  $\mathcal{D}_{f,g}$  of the holomorphic vector bundle  $(N_S^{\otimes \nu_{f,g}})^* \otimes TM|_S$  obtained by gluing together on  $S$  the local  $\mathfrak{D}_{f,g}$  defined in (1.7). We can also think to it as a holomorphic section of  $\text{Hom}(N_S^{\otimes \nu_{f,g}}, TM|_S)$ , that is as a morphism  $\mathcal{D}_{f,g} : N_S^{\otimes \nu_{f,g}} \rightarrow TM|_S$  of holomorphic vector bundles.

Summing up, if  $(z^1, \dots, z^n)$  are local coordinates adapted to  $(g, S)$  then  $\mathcal{D}_{f,g}$  is locally defined by

$$\mathcal{D}_{f,g} \stackrel{loc.}{=} \sum_{j=1}^n h^j|_S (\partial^* z^1)^\nu \otimes \frac{\partial}{\partial z^j} \Big|_S,$$

where the  $h^j$  are the ones appearing in (1.6). Observe that  $\mathcal{D}_{f,g}$  is not identically vanishing on  $S$  by construction.

**Definition 1.8.** A point  $p \in S$  is a *singularity* of  $\mathcal{D}_{f,g}$  if the associated morphism  $N_S^{\otimes \nu_{f,g}} \rightarrow TM|_S$  is not injective in  $p$ , that is if  $\mathcal{D}_{f,g}(p) = 0$ . We denote the *singular set* of  $\mathcal{D}_{f,g}$  by  $\text{Sing}(f, g)$ .

**Remark 1.9.** If we consider the couple  $(f, \text{Id}_M)$  we are in the setting of [3]. In this case every local adapted chart is clearly also  $\text{Id}_M$ -adapted, moreover Proposition 1.6 turns out to be [3, Prop.3.1.] and the canonical section  $\mathcal{D}_{f, \text{Id}_M}$  is exactly the canonical section  $X_f$  of [3, Def.3.2.].

## 2. THE CANONICAL FOLIATIONS

Let  $M$  and  $S$  be as in Section 1, with  $S' = S - \text{Sing}(S)$  the regular part of  $S$ , and let  $(f, g) \in \text{End}_S^2(M)$  be a couple whose order of coincidence is  $\nu = \nu_{f,g}$  and in which  $g$  is a local biholomorphism around  $S'$ . As just seen,  $(f, g)$  induces a canonical section  $\mathcal{D}_{f,g} : N_{S'}^{\otimes \nu} \rightarrow TM|_{S'}$  but we would like to have a 1-dimensional holomorphic (possibly singular) foliation on  $S'$ , that is an injective morphism  $\mathcal{F} : \mathcal{F} \rightarrow \mathcal{T}_{S'}$  of  $\mathcal{O}_{S'}$ -modules where  $\mathcal{F}$  is a rank 1 locally free  $\mathcal{O}_{S'}$ -module. Recall that its (possibly) singular set is

$$\text{Sing}(\mathcal{F}) = \{x \in S' \text{ s.t. } (\mathcal{T}_{S'}/\mathcal{F})_x \text{ is not a free } \mathcal{O}_{S',x}\text{-module}\}$$

Equivalently, a foliation on  $S'$  can be described as a morphism  $\mathcal{F} : F \rightarrow TS'$  of holomorphic vector bundles on  $S'$  where  $F$  is a line bundle. In this case its (possibly) singular set could be described as

$$\text{Sing}(\mathcal{F}) = \{x \in S' \text{ s.t. } \mathcal{F}(x) = 0\}.$$

**Definition 2.1.** The  $\mathcal{O}_{S'}$ -module  $\mathcal{F}$  is called *tangent sheaf of  $\mathcal{F}$* , while the holomorphic vector bundle  $F$  is the *tangent bundle of  $\mathcal{F}$* . Clearly  $\mathcal{F}$  is the sheaf of germs of holomorphic sections of  $F$ .

In the current section we will discuss two conditions under which we have a foliation (or more foliations) on  $S'$  induced by the canonical section  $\mathcal{D}_{f,g}$ . The first is a condition on the couple  $(f, g)$  while the second is on the way  $S'$  is embedded into  $M$ .

Let  $p \in S$  be any point. We have two induced ring homomorphisms given by pull-back of germs by  $f$  or  $g$ ,

$$\mathcal{O}_{M, f(p)} \xrightarrow{f_p^*, g_p^*} \mathcal{O}_{M, p}.$$

Clearly in general  $f_p^* \neq g_p^*$  but since  $f|_S = g|_S$  then  $(f_p^*)^{-1}(\mathcal{I}_{S, p}) = (g_p^*)^{-1}(\mathcal{I}_{S, p})$  and we denote this ideal of  $\mathcal{O}_{M, f(p)}$  by  $I_{f(p)}$ .

**Remark 2.2.** Observe that  $I_{f(p)} \supseteq \mathcal{I}_{f(S), f(p)}$  but they are not the same in general. Anyway  $I_{f(p)}$  may be seen as the stalk of a coherent module defined on an open neighborhood of  $f(p)$ , for every  $p \in S$ . In fact, by the hypothesis for any  $p \in S$  there is an open neighborhood  $U \subset M$  such that  $g : U \rightarrow g(U)$  is a biholomorphism and then  $f(U \cap S) = g(U \cap S)$  is an analytic sub-variety of  $g(U) \subset M$ . Hence  $\mathcal{I}_{f(U \cap S)}$  is a coherent  $\mathcal{O}_{g(U)}$ -module and  $I_{f(p)} = \mathcal{I}_{f(U \cap S), f(p)}$ .

**Definition 2.3.** Let  $p \in S$ . If

$$\min\{\nu_{f, g}^p(h), \text{ for } h \in I_{f(p)}\} > \nu_{f, g}^p$$

we say that  $(f, g)$  is *tangential at  $p \in S$* .

**Lemma 2.4.** *The following two statements are true:*

- i) *Let  $p \in S$  be any point and  $\{\rho^1, \dots, \rho^k\}$  any set of generators of  $I_{f(p)}$ . Then  $(f, g)$  is tangential at  $p$  if and only if*

$$\min\{\nu_{f, g}^p(\rho^1), \dots, \nu_{f, g}^p(\rho^k)\} > \nu_{f, g}^p.$$

- ii) *If  $(f, g)$  is tangential at a point  $p \in S$ , then it is tangential at all the points of  $S$ .*

*Proof.* i) Let  $h \in I_{f(p)}$ , then  $h = h_1 \rho^1 + \dots + h_k \rho^k$  for some  $h_1, \dots, h_k \in \mathcal{O}_{M, f(p)}$ . We can easily check that

$$h \circ f - h \circ g = \sum_{j=1}^k (h_j \circ f) (\rho^j \circ f - \rho^j \circ g) + \sum_{j=1}^k (\rho^j \circ g) (h_j \circ f - h_j \circ g).$$

The  $j$ -th term of the first sum is in  $\mathcal{I}_{S,p}^{\nu_{f,g}^p(\rho^j)}$  and each term of the second sum is in  $\mathcal{I}_{S,p}^{\nu_{f,g}^p+1}$ , so by definition

$$\nu_{f,g}^p(h) \geq \min\{\nu_{f,g}^p(\rho^1), \dots, \nu_{f,g}^p(\rho^k), \nu_{f,g}^p + 1\}.$$

This is true for any  $h \in I_{f(p)}$  hence

$$\min\{\nu_{f,g}^p(h), \text{ for } h \in I_{f(p)}\} \geq \min\{\nu_{f,g}^p(\ell^1), \dots, \nu_{f,g}^p(\ell^k), \nu_{f,g}^p + 1\}.$$

Reminding Definition 2.3 the assertion follows easily.

ii) By Remark 2.2 if  $\{\rho^1, \dots, \rho^k\}$  are generators of  $I_{f(p)}$  then the corresponding germs are generators of  $I_{f(q)}$  for all  $q \in S$  close enough to  $p$ . Then *ii*) of Lemma 1.2 and *i*) of the current Lemma imply that the set of points of  $S$  where  $f$  and  $g$  are tangential is open and closed at the same time and the assertion follows because  $S$  is connected.  $\square$

By Lemma 2.4 we can say that  $(f, g)$  is *tangential along*  $S$  if it is tangential at some point  $p \in S$ .

Now let assume in the following  $S$  non-singular for simplicity and let  $p$  be any point of  $S$ ,  $(z^1, \dots, z^n)$  any local coordinates adapted to  $(g, S)$  at  $p$  and  $(w^1, \dots, w^n)$  the corresponding special coordinates at  $f(p)$ . The ideal  $I_{f(p)}$  is clearly generated by the germ of  $w^1$  at  $f(p)$ , hence by Lemma 2.4 it easily follows that:

$$\begin{aligned} (f, g) \text{ tangential along } S &\iff h^1 \in \mathcal{I}_{S,p}, \text{ for every local coordinates} \\ &\quad \text{adapted to } (g, S) \text{ at } p, \text{ for every } p \in S \\ &\iff \mathcal{D}_{f,g} \text{ is in fact "tangential to } S" \text{ i.e.} \\ &\quad \text{a section of the bundle of } (N_S^{\otimes \nu})^* \otimes TS, \end{aligned}$$

where the  $h^1$  are the ones in (1.6). Therefore when  $(f, g)$  is tangential along  $S$  the canonical section  $\mathcal{D}_{f,g}$  is in fact a 1-dimensional holomorphic foliation on  $S$ ,

$$\mathcal{D}_{f,g} : N_S^{\otimes \nu} \longrightarrow TS,$$

and  $N_S^{\otimes \nu}$  is the tangent bundle of the foliation. Observe that the possibly singular set of  $\mathcal{D}_{f,g}$  is clearly  $\text{Sing}(f, g)$ . We can also think the foliation to be the distribution

$$\Xi_{f,g} = \mathcal{D}_{f,g}(N_S^{\otimes \nu}) \subset TS,$$

and if  $S^0 = S - \text{Sing}(f, g)$  then  $\Xi_{f,g}|_{S^0}$  is a line sub-bundle of  $TS^0$ . If  $(z^1, \dots, z^n)$  are any local coordinates adapted to  $(g, S)$  defined on an open subset  $U \subset M$  such that  $U \cap S \neq \emptyset$  then

$$X_{f,g} = \mathcal{D}_{f,g}((\partial z^1)^\nu) = \sum_{j=2}^n h^j|_S \frac{\partial}{\partial z^j} \Big|_S \quad (2.1)$$

is a local generator of  $\mathcal{D}_{f,g}$  (eq.  $\Xi_{f,g}$ ) on  $U \cap S$ , where the  $h^j$  are the ones appearing in (1.6). We call it a *canonical local generator* of  $\mathcal{D}_{f,g}$  (eq.  $\Xi_{f,g}$ ).

**Remark 2.5.** If we consider the couple  $(f, \text{Id}_M)$  then clearly it is tangential along  $S$  if and only if  $f$  is tangential along  $S$  according to [3, Def.1.2.]. Moreover our foliation  $\Xi_{f, \text{Id}_M}$  is exactly the foliation  $\Xi_f$  of [3, Def.3.2.] (when  $f$  is tangential along  $S$ ), which they call ‘canonical distribution’.

**Remark 2.6.** When  $(f, g)$  is tangential along  $S$  we would be able to define a 1-dimensional holomorphic foliation on  $S$  even weakening the hypothesis on  $g$ . If we only assume that  $f|_S = g|_S : S \rightarrow M$  is a local holomorphic embedding, that is

$$df|_{T_p S} = dg|_{T_p S} : T_p S \longrightarrow T_{f(p)} M$$

is injective for every  $p \in S$ , we are able to define a sort of “pre-canonical section”

$$\mathcal{D}_{f,g}^{pre} : N_S^{\otimes \nu} \longrightarrow (f^* TM)|_S.$$

By the hypothesis we have the injective morphism

$$D_S = df|_{TS} = dg|_{TS} : TS \longrightarrow (f^* TM)|_S$$

and one can prove that  $(f, g)$  is tangential along  $S$  (Remark 2.2 can be suitably adapted and then Lemma 2.4 is still true) if and only if  $\mathcal{D}_{f,g}^{pre}(N_S^{\otimes \nu}) \subset D_S(TS)$ . It follows that in this case we can define the 1-dimensional holomorphic foliation on  $S$

$$\mathcal{D}_{f,g} = D_S^{-1} \circ \mathcal{D}_{f,g}^{pre} : N_S^{\otimes \nu} \longrightarrow TS.$$

On the contrary, if  $(f, g)$  is not tangential along  $S$  we can not even define a morphism with image into  $TM|_S$  (that is the canonical section) and this is one of the reasons we assume the stronger hypothesis on  $g$  (the other is that anyway with the weaker hypothesis one can not extend locally the foliation  $\mathcal{D}_{f,g}$  arising in the tangential case - see Section 4 to understand what we mean).

If  $(f, g)$  is not tangential along  $S$  but  $S$  sits into  $M$  in a particularly nice way we still have 1-dimensional holomorphic foliations on  $S$ . For details about this property of  $S$  see [3, Sec.2] (or for a deeper treatment [4, Sec.2 and 3]), here we just recall some facts and remarks which we are going to use in this paper.

**Definition 2.7.** Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a complex sub-manifold of any codimension  $k$  ( $0 < k < n$ ). We say that  $S$  splits into  $M$  if the short exact sequence

$$0 \longrightarrow TS \xrightarrow{i} TM|_S \xrightarrow{\pi} N_S \longrightarrow 0$$

splits (holomorphically), that is if there exists a morphism  $\sigma : N_S \rightarrow TM|_S$  of holomorphic vector bundles such that  $\pi \circ \sigma = \text{Id}_{N_S}$ .

Let  $\mathfrak{U}$  be an atlas of  $M$  adapted to  $S$ . We call it a *splitting atlas adapted to  $S$*  if

$$\frac{\partial \hat{z}^i}{\partial z^j} \in \mathcal{I}_S(U \cap \hat{U}), \quad i = k+1, \dots, n, \quad j = 1, \dots, k, \quad (2.2)$$

for any two charts  $(U, z)$  and  $(\hat{U}, \hat{z})$  in  $\mathfrak{U}$  such that  $U \cap \hat{U} \neq \emptyset$ .

Then

**Proposition 2.8.** *Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a complex sub-manifold of any codimension  $k$  ( $0 < k < n$ ). The following statements are equivalent:*

- (i)  $S$  splits into  $M$
- (ii) there exists a morphism  $\tau : TM|_S \rightarrow TS$  of holomorphic vector bundles such that  $\tau \circ \iota = Id_{TS}$
- (iii) there exists a splitting atlas adapted to  $S$

*Proof.* (i)  $\iff$  (iii): see [3, Prop.2.1.] or even [4, Prop.2.15.].

(i)  $\iff$  (ii): suppose there exists  $\sigma$ , then  $\pi \circ (Id_{TM|_S} - \sigma \circ \pi) = 0$  by the property of  $\sigma$  and this implies that  $\text{im}(Id_{TM|_S} - \sigma \circ \pi) \subset \ker(\pi) = \text{im}(\iota)$ . Hence  $\tau = \iota^{-1} \circ (Id_{TM|_S} - \sigma \circ \pi)$  does the work.

Conversely, suppose there exists  $\tau$ . By its property it follows that  $\ker(\tau) \cap \text{im}(\iota) = \ker(\tau) \cap \ker(\pi) = \{\text{zero section}\}$  hence we can invert  $\pi|_{\ker(\tau)}$  and  $\sigma = \pi|_{\ker(\tau)}^{-1}$ .  $\square$

Observe that by definitions  $\tau \circ \sigma = \iota^{-1} \circ (\sigma - \sigma) = 0$ , hence  $\text{im}(\sigma) \subset \ker(\tau)$ . Moreover  $\ker(\tau) = \ker(Id_{TM|_S} - \sigma \circ \pi)$  then  $\text{im}(\sigma) = \ker(\tau)$ .

**Example 2.9.** Let  $S \subset M$  be a sub-manifold and  $U \subset M$  an open neighborhood of  $S$  in  $M$ . If there exists a holomorphic retraction  $\rho : U \rightarrow S$  then clearly  $S$  splits into  $M$ .

**Example 2.10.** A rank  $k$  holomorphic vector bundle  $\pi : M \rightarrow S$  is a holomorphic retraction of  $M$  on  $S$  (just identify  $S$  with the image of the zero section of the bundle). Thus the base of a holomorphic vector bundle can be seen as a splitting sub-manifold of the total space of the bundle. The local charts on  $M$  induced by a trivialization of the bundle clearly give a splitting atlas adapted to  $S$ .

Let  $S \subset M$  be a sub-manifold of any codimension  $k$  ( $0 < k < n$ ) and  $\sigma : N_S \rightarrow TM|_S$  any *splitting morphism* of  $S$  into  $M$ . If  $\mathfrak{U}$  is any splitting atlas adapted to  $S$  and  $(U, z) \in \mathfrak{U}$  any local chart then  $\{\partial z^1 = \pi(\frac{\partial}{\partial z^1}|_S), \dots, \partial z^k = \pi(\frac{\partial}{\partial z^k}|_S)\}$  is a local frame for  $N_S$  but in general  $\sigma(\partial z^j) \neq \frac{\partial}{\partial z^j}|_S$  for  $j = 1, \dots, k$  (we only know that their differences are in  $\mathcal{T}_S$ ). This observation leads to the following definition.

**Definition 2.11.** Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a complex sub-manifold of codimension  $k$  ( $0 < k < n$ ) which splits into  $M$ . Let  $\sigma$  be a splitting morphism. An atlas  $\mathfrak{U}$  of  $M$  is said to be a  $\sigma$ -*splitting atlas adapted to  $S$*  if it is a splitting atlas adapted to  $S$  such that  $\sigma(\partial z^j) = \frac{\partial}{\partial z^j}|_S$  for  $j = 1, \dots, k$ .

If moreover there is a local biholomorphism  $\varphi : W \rightarrow M$  defined on an open neighborhood  $W \subset M$  of  $S$  and every  $(U, z) \in \mathfrak{U}$  is such that  $\varphi|_U$  is a

biholomorphism onto its image, then we call  $\mathfrak{U}$  a  $\sigma$ -splitting atlas adapted to  $(\varphi, S)$ .

One can easily show that given  $S \subset M$  and a splitting morphism  $\sigma : N_S \rightarrow TM|_S$  then a  $\sigma$ -splitting atlas adapted to  $S$  always exists (hence if there is also a  $\varphi : W \rightarrow M$  as in Definition 2.11 a  $\sigma$ -splitting atlas adapted to  $(\varphi, S)$  always exists).

Now let  $S \subset M$  be of codimension 1 again and let  $(f, g) \in \text{End}_S^2(M)$  be a couple with  $g$  a local biholomorphism around  $S$ , whose order of coincidence is  $\nu = \nu_{f,g}$ . If  $S$  splits into  $M$  then for any splitting morphism  $\sigma$  we can define the 1-dimensional holomorphic foliation on  $S$

$$\mathcal{D}_{f,g}^\sigma = \tau^\sigma \circ \mathcal{D}_{f,g} : N_S^{\otimes \nu} \longrightarrow TS,$$

where  $\tau^\sigma = \iota^{-1} \circ (\text{Id}_{TM|_S} - \sigma \circ \pi)$ . The tangent bundle of this foliation is  $N_S^{\otimes \nu}$  and its (possibly) singular set  $\text{Sing}(\mathcal{D}_{f,g}^\sigma)$  may be larger than  $\text{Sing}(f, g)$ . As before, we can also think the foliation to be the distribution

$$\Xi_{f,g}^\sigma = \mathcal{D}_{f,g}^\sigma (N_S^{\otimes \nu}) \subset TS,$$

and if  $S^0 = S - \text{Sing}(\mathcal{D}_{f,g}^\sigma)$  then  $\Xi_{f,g}^\sigma|_{S^0}$  is a line sub-bundle of  $TS^0$ . If  $\mathfrak{U}$  is a  $\sigma$ -splitting atlas adapted to  $(g, S)$  and  $(U, z) \in \mathfrak{U}$  such that  $U \cap S \neq \emptyset$

$$\tau^\sigma \left( \frac{\partial}{\partial z^1} \Big|_S \right) = \tau^\sigma \circ \sigma (\partial z^1) = 0,$$

then

$$X_{f,g}^\sigma = \mathcal{D}_{f,g}^\sigma ((\partial z^1)^\nu) = \sum_{j=2}^n h^j|_S \frac{\partial}{\partial z^j} \Big|_S \quad (2.3)$$

is a local generator of  $\mathcal{D}_{f,g}^\sigma$  (eq.  $\Xi_{f,g}^\sigma$ ) on  $U \cap S$ , where the  $h^j$  are the ones appearing in (1.6). We call it a *canonical local generator* of  $\mathcal{D}_{f,g}^\sigma$  (eq.  $\Xi_{f,g}^\sigma$ ).

When  $\nu_{f,g} = 1$  we can define even other 1-dimensional holomorphic foliations on  $S$ . By the hypothesis on  $(f, g)$  we have the morphisms of holomorphic vector bundles on  $S$

$$df|_S, dg|_S : TM|_S \longrightarrow (f^*TM)|_S$$

which are different but coincide on  $TS \subset TM|_S$ . Since  $dg|_S$  is invertible we can compose  $dg|_S^{-1} \circ df|_S$  and it induces the morphism

$$d_{f,g} : N_S \longrightarrow N_S \quad (2.4)$$

$$[v] \longrightarrow [(dg_p^{-1} \circ df_p)(v)], \quad v \in T_p M, p \in S.$$

As a consequence we can define the 1-dimensional holomorphic foliations

$$\mathcal{D}_{f,g}^{\sigma,1} = \mathcal{D}_{f,g}^\sigma \circ d_{f,g} : N_S \longrightarrow TS$$

on  $S$ , where  $\sigma$  is any splitting morphism of  $S$  in  $M$  as before.

**Remark 2.12.** One might consider  $d_{f,g}^{\otimes \nu} : N_S^{\otimes \nu} \rightarrow N_S^{\otimes \nu}$  for any  $\nu \geq 1$  and define the foliations

$$\mathcal{D}_{f,g}^{\sigma,\nu} = \mathcal{D}_{f,g}^{\sigma} \circ d_{f,g}^{\otimes \nu} : N_S^{\otimes \nu} \longrightarrow TS$$

but since  $d_{f,g} = \text{Id}_{N_S}$  if and only if  $(f, g)$  is tangential along  $S$  or  $\nu > 1$  then it would be  $\mathcal{D}_{f,g}^{\sigma,\nu} = \mathcal{D}_{f,g}^{\sigma}$  for any  $\nu > 1$ .

The tangent bundle of this foliation is  $N_S$  and its (possibly) singular set  $\text{Sing}(\mathcal{D}_{f,g}^{\sigma,1})$  may be larger than  $\text{Sing}(\mathcal{D}_{f,g}^{\sigma})$ . As in the other cases we can also think the foliation to be the distribution

$$\Xi_{f,g}^{\sigma,1} = \mathcal{D}_{f,g}^{\sigma,1}(N_S) \subset TS$$

and if  $S^0 = S - \text{Sing}(\mathcal{D}_{f,g}^{\sigma,1})$  then  $\Xi_{f,g}^{\sigma,1}|_{S^0}$  is a line sub-bundle of  $TS^0$ . If  $\mathfrak{U}$  is a  $\sigma$ -splitting atlas adapted to  $(g, S)$  and  $(U, z) \in \mathfrak{U}$  such that  $U \cap S \neq \emptyset$  then one can easily check that

$$d_{f,g}(\partial z^1) = \left. \frac{\partial f^1}{\partial z^1} \right|_S \partial z^1 = (1 + h^1|_S) \partial z^1$$

where  $f^1 = w^1 \circ f$ , with  $(w^1, \dots, w^n)$  special coordinates induced by the coordinates  $z$  and  $h^1$  is the one appearing in (1.6). Hence

$$X_{f,g}^{\sigma,1} = \mathcal{D}_{f,g}^{\sigma}((1 + h^1|_S) \partial z^1) = (1 + h^1|_S) \sum_{j=2}^n h^j|_S \frac{\partial}{\partial z^j} \Big|_S \quad (2.5)$$

is a local generator of  $\mathcal{D}_{f,g}^{\sigma,1}$  (eq.  $\Xi_{f,g}^{\sigma,1}$ ) on  $U \cap S$ , where the  $h^j$  are the ones appearing in (1.6). We call it a *canonical local generator* of  $\mathcal{D}_{f,g}^{\sigma,1}$  (eq.  $\Xi_{f,g}^{\sigma,1}$ ).

**Remark 2.13.** Suppose that  $S$  splits into  $M$  and  $(f, g)$  is tangential along  $S$  at the same time and let  $\sigma$  be any splitting morphism. Since  $\tau^{\sigma} \circ \iota = \text{Id}_{TS}$  it follows that  $\mathcal{D}_{f,g}^{\sigma} = \mathcal{D}_{f,g}$  for any  $\sigma$  and  $\nu \geq 1$ . Moreover, if  $\nu = 1$  we have also  $\mathcal{D}_{f,g}^{\sigma,1} = \mathcal{D}_{f,g}$  by Remark 2.12.

We conclude by coming back to Example 2.10 for a while, in order to introduce a definition which we will use from Section 4 on.

If  $\pi : M \rightarrow S$  is a rank  $k$  holomorphic vector bundle then  $S$  is something more of a splitting sub-manifold of  $M$ . In fact, if  $(U, z)$  and  $(\hat{U}, \hat{z})$  are two local charts on  $M$  induced by a trivialization of the bundle, where  $(z^{k+1}, \dots, z^n)$  and  $(\hat{z}^{k+1}, \dots, \hat{z}^n)$  are local coordinates respectively on  $\pi(U) \subset S$  and  $\pi(\hat{U}) \subset S$ , then

$$\frac{\partial \hat{z}^i}{\partial z^j} \equiv 0 \quad \text{and} \quad \frac{\partial^2 \hat{z}^r}{\partial z^s \partial z^t} \equiv 0$$

for  $i = k+1, \dots, n$ ,  $j = 1, \dots, k$  and  $r, s, t = 1, \dots, k$ . This observation leads to the following definitions.



**Definition 2.14.** Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a complex sub-manifold of codimension  $k$  ( $0 < k < n$ ). We say that  $S$  is *comfortably embedded into  $M$*  if there exists a splitting atlas  $\mathfrak{U}$  adapted to  $S$  (hence  $S$  splits into  $M$ ) such that

$$\frac{\partial^2 \hat{z}^r}{\partial z^s \partial \hat{z}^t} \in \mathcal{I}_S(U \cap \hat{U}), \quad r, s, t = 1, \dots, k, \quad (2.6)$$

for any two charts  $(U, z)$  and  $(\hat{U}, \hat{z})$  in  $\mathfrak{U}$  such that  $U \cap \hat{U} \neq \emptyset$ .

Such an atlas is said to be a *comfortably atlas adapted to  $S$* . If moreover  $\sigma$  is a splitting morphism of  $S$  in  $M$  and  $\mathfrak{U}$  is also a  $\sigma$ -splitting atlas adapted to  $S$  then it is called a  *$\sigma$ -comfortably atlas adapted to  $S$* . Finally, if there is a local biholomorphism  $\varphi : W \rightarrow M$  defined on an open neighborhood  $W \subset M$  of  $S$  and every  $(U, z) \in \mathfrak{U}$  is such that  $\varphi|_U$  is a biholomorphism onto its image, then we call  $\mathfrak{U}$  a  *$\sigma$ -comfortably atlas adapted to  $(\varphi, S)$* .

Roughly speaking, a comfortably embedded sub-manifold is a sort of first-order approximation of the zero section of a holomorphic vector bundle.

**Example 2.15.** Let  $M$  be a  $n$ -dimensional complex manifold and  $p \in M$  some point. Let  $\pi : \tilde{M} \rightarrow M$  denote the blow-up of  $M$  at  $p$  and let  $S = \pi^{-1}(p) \cong \mathbb{P}^{n-1}$  be the *exceptional divisor*, which is a non-singular compact connected hypersurface in  $\tilde{M}$ . Then it is easy to check that  $S$  is comfortably embedded into  $\tilde{M}$ .

**Remark 2.16.** In both Section 1 and 2 we have assumed  $S \subset M$  to be a hypersurface, except for some general definitions. Anyway all definitions, lemmas, propositions and computations stated and done up to now may be easily adapted for  $S$  of any codimension  $k$  ( $0 < k < n$ ), likewise in the first three sections of [3]. Conversely, we need  $S$  to have codimension 1 for what follows.

### 3. A BAUM-BOTT-TYPE INDEX THEOREM

Let  $S$  be a  $m$ -dimensional complex manifold and suppose to have a 1-dimensional holomorphic (possibly singular) foliation

$$\mathcal{F} : F \rightarrow TS$$

on it. Set  $S^0 = S - \text{Sing}(\mathcal{F})$  and  $F^0 = F|_{S^0}$ , which we can identify with the line sub-bundle  $\mathcal{F}(F^0) \subset TS^0$ .

**Definition 3.1.** The quotient  $N_{\mathcal{F}} = TS^0/F^0$  is called *normal bundle of  $\mathcal{F}$* , while the virtual bundle (in the sense of  $K$ -theory)  $TS - F$  is the *virtual normal bundle of  $\mathcal{F}$* , since  $(TS - F)|_{S^0} = N_{\mathcal{F}}$  in the  $K$ -group of  $S^0$ .

If we express the foliation through the injective morphism  $\mathcal{F} : \mathcal{F} \rightarrow \mathcal{T}_S$  then the coherent  $\mathcal{O}_S$ -module  $\mathcal{N}_{\mathcal{F}} = \mathcal{T}_S/\mathcal{F}$  is called *normal sheaf of  $\mathcal{F}$* . Clearly  $\mathcal{F}^0 = \mathcal{F}|_{S^0}$  is the sheaf of germs of holomorphic sections of  $F^0$  and  $\mathcal{N}_{\mathcal{F}}^0 = \mathcal{N}_{\mathcal{F}}|_{S^0}$  is the sheaf of germs of holomorphic sections of  $N_{\mathcal{F}}$ .

Let  $\pi : TS^0 \rightarrow N_{\mathcal{F}}$  be the obvious projection. There is the natural partial holomorphic connection (in the sense of Bott [7]) on  $N_{\mathcal{F}}$  along  $F^0$

$$\begin{aligned} \mathcal{N}_{\mathcal{F}}^0 &\xrightarrow{\delta^{bb}} (\mathcal{F}^0)^* \otimes \mathcal{N}_{\mathcal{F}}^0 \\ w &\longrightarrow \delta^{bb}(w) \text{ s.t. } \delta^{bb}(w)(v) = \pi([v, \tilde{w}]), \end{aligned} \quad (3.1)$$

for any  $w \in \mathcal{N}_{\mathcal{F}}^0$  and  $v \in \mathcal{F}^0$ , where  $\tilde{w} \in \mathcal{T}_{S^0}$  is any vector field such that  $\pi(\tilde{w}) = w$ . Observe that  $\delta^{bb}(w)(v)$  is holomorphic whenever  $w$  and  $v$  are, and (3.1) is independent by the choice of  $\tilde{w}$ .

This partial connection makes  $N_{\mathcal{F}}$  a  $F^0$ -bundle (using the terminology of [7], see also [20, Sec.II.9.]) and as a consequence of the ‘Bott vanishing theorem’ (see [20, Th.II.9.11.] for a complete version or [11, Th.6.2.3.] for a simplified one) and of the exact sequence

$$0 \longrightarrow F^0 \xrightarrow{\mathcal{F}|_{S^0}} TS^0 \xrightarrow{\pi} N_{\mathcal{F}} \longrightarrow 0,$$

one can localize at  $\text{Sing}(\mathcal{F})$  suitable characteristic classes of the virtual normal bundle  $TS - F$ . In particular if  $S$  is *compact* one gets theorem [6, Th.1.] (see also [20, Th.III.7.6.] or [11, Th.6.2.5.]). Then by it and by Section 2 we gain the following index theorem.

**Theorem 3.2** (Baum-Bott-type index theorem). *Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a non-singular compact connected complex hypersurface in  $M$ . Let  $(f, g) \in \text{End}_S^2(M)$  be a couple where  $g$  is a local biholomorphism around  $S$  and set  $\nu = \nu_{f,g}$ . Assume that*

(i)  $(f, g)$  is tangential along  $S$

or that

(ii)  $S$  splits into  $M$ .

In case (i) set  $\mathcal{D} = \mathcal{D}_{f,g}$  while in case (ii) set  $\mathcal{D} = \mathcal{D}_{f,g}^\sigma$  (where  $\sigma$  is some splitting morphism) and suppose moreover that  $\mathcal{D} \neq 0$  and  $\mathcal{D}_{f,g}^{\sigma,1} \neq 0$ . Let  $\text{Sing}(\mathcal{D}) = \sqcup_\lambda \Sigma_\lambda$  and  $\text{Sing}(\mathcal{D}_{f,g}^{\sigma,1}) = \sqcup_\mu \Sigma_\mu^1$  be the decompositions in connected components of the singular sets of the foliations.

Then for any symmetric homogeneous polynomial  $\varphi \in \mathbb{C}[z_1, \dots, z_{n-2}]$  of degree  $n-1$  there exist complex numbers  $\text{Res}_\varphi(\mathcal{D}; TS - N_S^{\otimes \nu}; \Sigma_\lambda)$  and  $\text{Res}_\varphi(\mathcal{D}_{f,g}^{\sigma,1}; TS - N_S; \Sigma_\mu^1)$  such that

$$\sum_\lambda \text{Res}_\varphi(\mathcal{D}; TS - N_S^{\otimes \nu}; \Sigma_\lambda) = \int_S \varphi(TS - N_S^{\otimes \nu})$$

and, only when  $\nu = 1$ ,

$$\sum_\mu \text{Res}_\varphi(\mathcal{D}_{f,g}^{\sigma,1}; TS - N_S; \Sigma_\mu^1) = \int_S \varphi(TS - N_S).$$

We call Theorem 3.2 a *Baum-Bott-type index theorem* because Baum and Bott have introduced this kind of residues and the partial connection (3.1)

(see [6] and [5]). Moreover, as we said, the theorem follows by the Baum-Bott index theorem [6, Th.1.].

**Remark 3.3.** To be precise in case (ii) one just needs that  $S - \text{Sing}(f, g)$  splits into  $M$  since  $\text{Sing}(f, g) \subset \text{Sing}(\mathcal{D}_{f,g}^\sigma)$ .

**Remark 3.4.** We can think the foliations of Theorem 3.2 in terms of morphisms of  $\mathcal{O}_S$ -modules, that is as

$$\mathcal{D} : \mathcal{N}_S^{\otimes \nu} \rightarrow \mathcal{T}_S \quad \text{and} \quad \mathcal{D}_{f,g}^{\sigma,1} : \mathcal{N}_S \rightarrow \mathcal{T}_S,$$

which are injective morphisms. Recall that the coherent  $\mathcal{O}_S$ -modules  $\mathcal{N}_{\mathcal{D}} = \mathcal{T}_S / \mathcal{D}(\mathcal{N}_S^{\otimes \nu})$  and  $\mathcal{N}_{\mathcal{D}}^1 = \mathcal{T}_S / \mathcal{D}_{f,g}^{\sigma,1}(\mathcal{N}_S)$  are the normal sheaves of  $\mathcal{D}$  and  $\mathcal{D}_{f,g}^{\sigma,1}$  and their restrictions to  $S^0$  are the sheaves of germs of holomorphic sections respectively of  $N_{\mathcal{D}}$  and  $N_{\mathcal{D}_{f,g}^{\sigma,1}}$ . Then we have

$$\varphi(TS - N_S^{\otimes \nu}) = \varphi(\mathcal{N}_{\mathcal{D}}) \quad \text{and} \quad \varphi(TS - N_S) = \varphi(\mathcal{N}_{\mathcal{D}}^1)$$

for any symmetric homogeneous polynomial  $\varphi$  (see [20, Ch.VI]).

**Remark 3.5.** If we consider the couple  $(f, \text{Id}_M)$  then Theorem 3.2 turns out to be [3, Th.6.4.]. Observe that Abate-Bracci-Tovena assume  $S - \text{Sing}(f, \text{Id}_M)$  to be comfortably embedded into  $M$  and not only splitting, however the splitting property is sufficient. Moreover they do not consider the foliation  $\mathcal{D}_{f,\text{Id}}^\sigma$  when  $\nu = 1$  but only  $\mathcal{D}_{f,\text{Id}}^{\sigma,1}$  (which in their notation are respectively  $H_{\sigma,f}$  and  $H_{\sigma,f}^1$ ).

**Remark 3.6.** By Remark 2.6 when  $(f, g)$  is tangential along  $S$  we may assume only that  $f|_S = g|_S : S \rightarrow M$  is a local holomorphic embedding (instead of the stronger “ $g$  is a local biholomorphism around  $S$ ”) and get anyway an index theorem likewise Theorem 3.2 yet.

We conclude this section by deriving explicit formulas for the computation of the residues in Theorem 3.2 at isolated singular points. For this purpose we briefly recall how these residues are defined in Lehmann-Suwa theory. We do it considering only foliations  $\mathcal{D}$  for simplicity, anyway for  $\mathcal{D}_{f,g}^{\sigma,1}$  it is the same. As a reference consult [20, Sec.III.7.].

Let  $\varphi \in \mathbb{C}[z_1, \dots, z_{n-2}]$  be any symmetric homogeneous polynomial of degree  $n - 1$  and  $\delta_{\mathcal{D}}^{bb}$  be the partial holomorphic connection on  $N_{\mathcal{D}}$  along  $N_{S^0}^{\otimes \nu}$  defined as in (3.1). Let  $\nabla_{\mathcal{D}}^{bb}$  be any  $(1, 0)$ -type extension of it, which always exists (see for example [6, Lem.2.5.]). Then put connections  $\nabla_1^0$  and  $\nabla_2^0$  respectively on  $N_{S^0}^{\otimes \nu}$  and  $TS^0$  such that the triple  $(\nabla_1^0, \nabla_2^0, \nabla_{\mathcal{D}}^{bb})$  is compatible (see [20, p.72] for the meaning) with the short exact sequence

$$0 \longrightarrow N_{S^0}^{\otimes \nu} \xrightarrow{\mathcal{D}|_{S^0}} TS^0 \xrightarrow{\pi} N_{\mathcal{D}} \longrightarrow 0 \quad (3.2)$$

and set  $\nabla_{\mathcal{D}}^\bullet = (\nabla_1^0, \nabla_2^0)$ . One makes this choices since by the compatibility of the triple and the Bott vanishing theorem

$$\varphi(\nabla_{\mathcal{D}}^\bullet) = \varphi\left(\nabla_{\mathcal{D}}^{bb}\right) = 0,$$

where  $\varphi(\nabla_{\mathcal{D}}^{\bullet})$  is a  $2(n-1)$  form on  $S^0$  defined as at [20, pp.71-72]. Let now  $\Sigma$  be a connected component of  $\text{Sing}(\mathcal{D})$  and  $V \subset S$  an open sub-set such that  $V \cap \text{Sing}(\mathcal{D}) = \Sigma$ . Choose any connections  $\nabla_1^V$  and  $\nabla_2^V$  on  $N_S^{\otimes \nu}|_V$  and  $TS|_V$  and set  $\nabla_V^{\bullet} = (\nabla_1^V, \nabla_2^V)$ . Finally, let  $R \subset V$  be a compact real sub-manifold of dimension  $2(n-1)$  oriented as  $S$  and such that  $\Sigma \subset \text{int}(R)$ . Consider on the boundary  $\partial R$  the orientation induced by  $R$ . Then by definition the residue is

$$\text{Res}_{\varphi}(\mathcal{D}; TS - N_S^{\otimes \nu}; \Sigma) = \int_R \varphi(\nabla_V^{\bullet}) - \int_{\partial R} \varphi(\nabla_{\mathcal{D}}^{\bullet}, \nabla_V^{\bullet}),$$

where the Bott difference form  $\varphi(\nabla_{\mathcal{D}}^{\bullet}, \nabla_V^{\bullet})$  is a  $2(n-1) - 1$  form on  $V - \Sigma$  defined as at [20, pp.71-72]. One can show that this formula does not depend on the choice of the various connections or of the sub-manifold  $R$ .

If  $\Sigma = \{p\}$  is an isolated point we can assume  $V$  to be such that  $N_S^{\otimes \nu}|_V$  and  $TS|_V$  are trivial. Hence if we take  $\nabla_1^V$  and  $\nabla_2^V$  trivial respect to some local frames  $\varphi(\nabla_V^{\bullet}) = 0$  and the residue becomes

$$\text{Res}_{\varphi}(\mathcal{D}; TS - N_S^{\otimes \nu}; p) = - \int_{\partial R} \varphi(\nabla_{\mathcal{D}}^{\bullet}, \nabla_V^{\bullet}).$$

Observe now that  $N_{S^0}^{\otimes \nu}$  and  $TS^0$  are not in general  $N_{S^0}^{\otimes \nu}$ -bundles but if there is a local generator  $X$  of the foliation  $\mathcal{D}$  on  $V$  then

$$N_{S^0}^{\otimes \nu}|_{V-\{p\}} \quad \text{and} \quad TS^0|_{V-\{p\}}$$

are canonically  $N_{S^0}^{\otimes \nu}|_{V-\{p\}}$ -bundles thanks to the natural ‘holomorphic action’ of  $X$  on them by Lie bracket  $[X, \cdot]$  (again we are using the terminology of [20, Sec.II.9.]). This action induces partial holomorphic connections on  $N_{S^0}^{\otimes \nu}|_{V-\{p\}}$  and  $TS^0|_{V-\{p\}}$  along  $N_{S^0}^{\otimes \nu}|_{V-\{p\}}$  which are, together with  $\delta_{\mathcal{D}}^{bb}$ , compatible with (3.2) restricted to  $V - \{p\}$ . Therefore we can assume  $\nabla_1^0$  and  $\nabla_2^0$  to be ‘ $X$ -connections’ defined on  $V - \{p\}$  (and again such that the triple  $(\nabla_1^0, \nabla_2^0, \nabla_{\mathcal{D}}^{bb})$  is compatible with (3.2)). With this choice one can show that

$$\text{Res}_{\varphi}(\mathcal{D}; TS - N_S^{\otimes \nu}; p) = - \int_{\partial R} \varphi(\nabla_2^0, \nabla_2^V)$$

and using the same arguments for  $\mathcal{D}_{f,g}^{\sigma,1}$  also that

$$\text{Res}_{\varphi}(\mathcal{D}_{f,g}^{\sigma,1}; TS - N_S; p) = - \int_{\partial R} \varphi(\nabla_2^0, \nabla_2^V),$$

where  $\nabla_2^0$  and  $\nabla_2^V$  are as above.

Now one can work as in the proof of [20, Th.III.5.5.] (see also [15, Sec.5.]) and obtain a similar formula. In particular, let  $(U, z)$  be a local chart of  $M$  at  $p$  belonging to an atlas  $\mathfrak{U}$  adapted to  $(g, S)$  (or a  $\sigma$ -splitting atlas adapted to  $(g, S)$ , if necessary), set  $V = U \cap S$  and let  $X$  be the canonical local generators (2.1), (2.3) or (2.5) (depending on the case). Moreover let  $\nabla_2^V$  be trivial respect to the local frame  $\{\frac{\partial}{\partial z^2}|_V, \dots, \frac{\partial}{\partial z^n}|_V\}$  of  $TS$  and let the  $h^j \in \mathcal{O}_{M,p}$  be the ones appearing in (1.6). Then when  $(f, g)$  is tangential

along  $S$  and  $\mathcal{D} = \mathcal{D}_{f,g}$  or when  $S$  splits into  $M$  and  $\mathcal{D} = \mathcal{D}_{f,g}^\sigma$  we get the formula

$$\text{Res}_\varphi(\mathcal{D}; TS - N_S^{\otimes \nu}; p) = \int_\Gamma \frac{\varphi(-H)}{h^2 \dots h^n} dz^2 \wedge \dots \wedge dz^n, \quad (3.3)$$

where

$$H = \left( \frac{\partial h^j}{\partial z^k} \Big|_{U \cap S} \right)_{j,k=2,\dots,n}$$

and  $\Gamma$  is the  $(n-1)$  cycle

$$\Gamma = \{q \in U \cap S \text{ s.t. } |h^2(q)| = \dots = |h^n(q)| = \epsilon\},$$

for  $\epsilon > 0$  small enough, oriented so that  $d\vartheta^2 \wedge \dots \wedge d\vartheta^n > 0$  where  $\vartheta^j = \arg(h^j)$ . Similarly, when  $S$  splits into  $M$  and  $\nu = 1$  we have

$$\text{Res}_\varphi(\mathcal{D}_{f,g}^{\sigma,1}; TS - N_S; p) = \int_{\Gamma'} \frac{\varphi(-H')}{(1+h^1)^{n-1} h^2 \dots h^n} dz^2 \wedge \dots \wedge dz^n, \quad (3.4)$$

where

$$H' = \left( \frac{\partial(1+h^1)h^j}{\partial z^k} \Big|_{U \cap S} \right)_{j,k=2,\dots,n}$$

and

$$\Gamma' = \{q \in U \cap S \text{ s.t. } |(1+h^1(q))h^2(q)| = \dots = |(1+h^1(q))h^n(q)| = \epsilon\},$$

for  $\epsilon > 0$  small enough, oriented so that  $d\vartheta^2 \wedge \dots \wedge d\vartheta^n > 0$  where  $\vartheta^j = \arg(1+h^1)(h^j)$ .

**Remark 3.7.** Implicitly we have proved (and used) that

$$\text{Res}_\varphi(\mathcal{D}; TS - N_S^{\otimes \nu}; p) = \text{Res}_\varphi(X; TS; p)$$

and

$$\text{Res}_\varphi(\mathcal{D}_{f,g}^{\sigma,1}; TS - N_S; p) = \text{Res}_\varphi(X; TS; p),$$

where  $X$  is a local generator of the foliations  $\mathcal{D}$  or  $\mathcal{D}_{f,g}^{\sigma,1}$  and the residues on the right are the ones associated to the natural ‘holomorphic action’ of  $X$  on  $TS^0|_{V-\{p\}}$  by Lie bracket (see [20, Sec.III.5.] for definition). Consequently we could have used directly the formula of [20, Th.III.5.5.]. See also [20, Rmk.III.7.7.(1)] or [11, Rmk.6.2.2(1)].

## 4. LOCAL EXTENSIONS OF THE FOLIATIONS

Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  an analytic subvariety of pure dimension  $m$  with regular part  $S' = S - \text{Sing}(S)$ . Suppose to have a 1-dimensional holomorphic (possibly singular) foliation on  $S'$

$$\mathcal{F} : F \rightarrow TS'$$

leaving an extension to an open neighborhood  $W \subset M$  of  $S$ , that is there exists a 1-dimensional holomorphic foliation

$$\tilde{\mathcal{F}} : \tilde{F} \rightarrow TM|_W$$

on  $W$  such that  $\tilde{\mathcal{F}}|_{S'} \equiv \mathcal{F}$  (hence  $\tilde{\mathcal{F}}$  leaves  $S$  invariant). Let denote  $S^0 = S' - \text{Sing}(\mathcal{F})$  and  $F^0 = F|_{S^0}$  (which again is identified with  $\mathcal{F}(F^0) \subset TS^0$ ).

**Definition 4.1.** The quotient  $N_{\mathcal{F}}^M = TM|_{S^0} / F^0$  is called *normal bundle of  $\mathcal{F}$  in  $M$* . Observe that  $N_{\mathcal{F}}^M = N_{\tilde{\mathcal{F}}}^M|_{S^0}$  where  $N_{\tilde{\mathcal{F}}}^M$  is the normal bundle of  $\tilde{\mathcal{F}}$  as in Definition 3.1, defined on  $W - \text{Sing}(\tilde{\mathcal{F}})$ .

In this situation there are natural partial holomorphic connections along  $F^0$  on the normal bundle of  $S^0$  in  $M$ , namely  $N_{S^0} = TM|_{S^0} / TS^0$ , and on  $N_{\mathcal{F}}^M$ . Indeed, if  $\pi : TM|_{S^0} \rightarrow N_{S^0}$  is the obvious projection one can define

$$\begin{aligned} \mathcal{N}_{S^0} &\xrightarrow{\delta^{cs}} (\mathcal{F}^0)^* \otimes \mathcal{N}_{S^0} \\ w &\longrightarrow \delta^{cs}(w) \text{ s.t. } \delta^{cs}(w)(v) = \pi([\tilde{v}, \tilde{w}]|_{S^0}) \end{aligned} \quad (4.1)$$

for any  $w \in \mathcal{N}_{S^0}$  and  $v \in \mathcal{F}^0$ , where  $\tilde{w} \in \mathcal{T}_M|_{S^0}$  and  $\tilde{v} \in \tilde{\mathcal{F}}|_{S^0}$  are any vector fields such that  $\pi(\tilde{w}|_{S^0}) = w$  and  $\tilde{v}|_{S^0} = v$ . Observe that  $\delta^{cs}(w)(v)$  is holomorphic whenever  $w$  and  $v$  are and (4.1) is independent by the choices of  $\tilde{w}$  and  $\tilde{v}$ . Similarly, if  $\rho : TM|_{S^0} \rightarrow N_{\mathcal{F}}^M$  is the other obvious projection and  $\mathcal{O}(-)$  denotes the sheaves of germs of holomorphic sections of bundles, one can define

$$\begin{aligned} \mathcal{O}(N_{\mathcal{F}}^M) &\xrightarrow{\delta^{ls}} (\mathcal{F}^0)^* \otimes \mathcal{O}(N_{\mathcal{F}}^M) \\ w &\longrightarrow \delta^{ls}(w) \text{ s.t. } \delta^{ls}(w)(v) = \rho([\tilde{v}, \tilde{w}]|_{S^0}) \end{aligned} \quad (4.2)$$

for any  $w \in \mathcal{O}(N_{\mathcal{F}}^M)$  and  $v \in \mathcal{F}^0$ , where  $\tilde{w} \in \mathcal{T}_M|_{S^0}$  and  $\tilde{v} \in \tilde{\mathcal{F}}|_{S^0}$  are any sections such that  $\rho(\tilde{w}|_{S^0}) = w$  and  $\tilde{v}|_{S^0} = v$ . Again  $\delta^{ls}(w)(v)$  is holomorphic whenever  $w$  and  $v$  are and (4.2) is independent by the choices of  $\tilde{w}$  and  $\tilde{v}$ .

These partial connections, possibly with additional hypotheses on  $S$ , allow one to localize at  $\text{Sing}(S) \cup \text{Sing}(\mathcal{F})$  suitable characteristic classes of some holomorphic vector bundles defined in a neighborhood of  $S$  and then to get index theorems when  $S$  is compact. (see [20, Sec.IV.5 and IV.6.] or [11, Sec.6.3.2 and 6.3.3] for details).

**Remark 4.2.** In the setting just described the double inclusion  $F^0 \subset TS^0 \subset TM|_{S^0}$  induces the “normal exact sequence”

$$0 \longrightarrow N_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}^M \longrightarrow N_{S^0} \longrightarrow 0$$

and the canonical partial holomorphic connections along  $F^0$  defined in (3.1), (4.1) and (4.2) are compatible with it (see [20, p.72] for the meaning).

Let  $S$  be a globally irreducible hypersurface and  $(f, g) \in \text{End}_S^2(M)$  a couple where  $g$  is a local biholomorphism around  $S'$ . If  $(f, g)$  is tangential along  $S$  or if  $S'$  splits into  $M$  then we know by Section 2 that there are natural foliations on  $S'$ , namely  $\mathcal{D}_{f,g}$ ,  $\mathcal{D}_{f,g}^\sigma$  and  $\mathcal{D}_{f,g}^{\sigma,1}$ . In general they cannot be extended to foliations on a whole neighborhood of  $S$  but we can do something weaker. Indeed we can extend them *locally* (about points of  $S'$ ) in a quite good way, that is so that we are able to define partial holomorphic connections almost as in (4.1) and (4.2). We are going to talk about this sort of “first order extensions” of foliations here, while the resulting partial holomorphic connections will be discussed in Section 5 and 6 (see [9] to investigate further this topic).

For simplicity, in the following let  $S$  be a non-singular hypersurface. Assume  $(f, g)$  to be tangential along  $S$  with order of coincidence  $\nu = \nu_{f,g}$  and fix an atlas  $\mathfrak{U}$  on  $M$  adapted to  $(g, S)$ . Recall that if  $(U, z) \in \mathfrak{U}$  is a local chart such that  $U \cap S \neq \emptyset$  then (2.1) gives the canonical local generator  $X_{f,g}$  of the foliation  $\mathcal{D}_{f,g}$  (eq.  $\Xi_{f,g}$ ) associated to these coordinates. We now define the local holomorphic vector field

$$\mathcal{X}_{f,g} = \mathfrak{D}_{f,g} \left( \left( \frac{\partial}{\partial z^1} \right)^{\otimes \nu} \right) = \sum_{j=1}^n h^j \frac{\partial}{\partial z^j}, \quad (4.3)$$

where  $\mathfrak{D}_{f,g}$  is defined in (1.7), to be the *canonical local extension* of  $X_{f,g}$ . Observe that it generates a 1-dimensional holomorphic foliation on  $U$  leaving  $U \cap S$  invariant, which restricted to  $U \cap S$  coincides with  $\mathcal{D}_{f,g}$ . Let now  $(\hat{U}, \hat{z}) \in \mathfrak{U}$  be another local chart such that  $U \cap \hat{U} \cap S \neq \emptyset$ ,  $\hat{X}_{f,g}$  the corresponding canonical local generator of  $\mathcal{D}_{f,g}$  on  $\hat{U} \cap S$  defined as in (2.1) and  $\hat{\mathcal{X}}_{f,g}$  its canonical local extension as in (4.3). Being both the charts adapted to  $S$  there exists a germ  $a \in \mathcal{O}_M^*$  such that  $\hat{z}^1 = az^1$ . Then one can easily check that

$$\hat{\mathcal{X}}_{f,g} = \left( \frac{1}{a|_S} \right)^\nu X_{f,g} \quad (4.4)$$

where they overlap. Instead the relation between their extensions is a bit more complicated and we describe it in the next proposition. First, we introduce the following notation:

- (1)  $T_k$  for holomorphic vector fields of the kind  $\sum_{j=2}^n b^j \frac{\partial}{\partial z^j}$ , with  $b^j \in \mathcal{I}_S^k$  (*tangential terms vanishing on  $S$  with ‘order  $k$ ’*);
- (2)  $V_k$  for holomorphic vector fields of the kind  $\sum_{j=1}^n c^j \frac{\partial}{\partial z^j}$ , with  $c^j \in \mathcal{I}_S^k$  (*generic terms vanishing on  $S$  with ‘order  $k$ ’*);

**Proposition 4.3.** *Let  $M$  be a  $n$ -dimensional complex manifold,  $S \subset M$  a non-singular connected complex hypersurface,  $(f, g) \in \text{End}_S^2(M)$  such that  $g$  is a local biholomorphism around  $S$  and set  $\nu = \nu_{f,g}$ .*

Suppose  $(f, g)$  tangential along  $S$  and let  $\mathfrak{U}$  be an atlas adapted to  $(g, S)$ . If  $(U, z)$  and  $(\hat{U}, \hat{z})$  are two local charts in  $\mathfrak{U}$  such that  $U \cap \hat{U} \cap S \neq \emptyset$  and  $\mathcal{X}_{f,g}$  and  $\hat{\mathcal{X}}_{f,g}$  are the corresponding local holomorphic vector fields defined as in (4.3), then

$$\hat{\mathcal{X}}_{f,g} = \left(\frac{1}{a}\right)^\nu \mathcal{X}_{f,g} + T_\nu + V_2$$

where they overlap, with  $a \in \mathcal{O}_M^*$  such that  $\hat{z}^1 = az^1$ .

*Proof.* Since  $\hat{z}^1 = az^1$  then

$$\frac{\partial}{\partial \hat{z}^1} = \left(\frac{1}{a} + \frac{\partial \frac{1}{a}}{\partial \hat{z}^1} \hat{z}^1\right) \frac{\partial}{\partial z^1} + \sum_{p=2}^n \frac{\partial z^p}{\partial \hat{z}^1} \frac{\partial}{\partial z^p}, \quad (4.5)$$

while

$$\frac{\partial}{\partial \hat{z}^j} = \frac{\partial \frac{1}{a}}{\partial \hat{z}^j} \hat{z}^1 \frac{\partial}{\partial z^1} + \sum_{p=2}^n \frac{\partial z^p}{\partial \hat{z}^j} \frac{\partial}{\partial z^p}, \quad j = 2, \dots, n. \quad (4.6)$$

By (1.6) we have that

$$\begin{aligned} \hat{h}^1 a^\nu (z^1)^\nu &= \hat{h}^1 (\hat{z}^1)^\nu = \hat{w}^1 \circ f - \hat{w}^1 \circ g = \\ &= (g_* a \circ f)(w^1 \circ f) - (g_* a \circ g)(w^1 \circ g) = \\ &= (w^1 \circ f)(g_* a \circ f - g_* a \circ g) + (g_* a \circ g) h^1 (z^1)^\nu, \end{aligned}$$

where  $w, \hat{w}$  are the special coordinates induced by respectively  $z, \hat{z}$  and we set  $g_* a = a \circ g^{-1}$ . Then by (1.1), (1.6) and observing that  $w^1 \circ f \in \mathcal{I}_S$

$$\hat{h}^1 a^\nu (z^1)^\nu = (w^1 \circ f)(z^1)^\nu \sum_{j=1}^n h^j \frac{\partial a}{\partial z^j} + a h^1 (z^1)^\nu \pmod{\mathcal{I}_S^{2\nu+1}}$$

since  $\frac{\partial g_* a}{\partial w^j} \circ g = \frac{\partial a}{\partial z^j}$ . Being  $(f, g)$  tangential along  $S$  it follows that

$$w^1 \circ f = w^1 \circ g \pmod{\mathcal{I}_S^{\nu+1}} = z^1 \pmod{\mathcal{I}_S^{\nu+1}},$$

then we can improve (1.10) in

$$\hat{h}^1 a^\nu = z^1 \sum_{j=2}^n h^j \frac{\partial a}{\partial z^j} + a h^1 \pmod{\mathcal{I}_S^2}. \quad (4.7)$$

Finally, observe that

$$\delta_{ij} = \frac{\partial z^i}{\partial z^j} = \sum_{k=1}^n \frac{\partial z^i}{\partial \hat{z}^k} \frac{\partial \hat{z}^k}{\partial z^j}, \quad i, j = 1, \dots, n. \quad (4.8)$$

Then manipulating  $\hat{\mathcal{X}}_{f,g} = \sum_{j=1}^n \hat{h}^j \frac{\partial}{\partial \hat{z}^j}$  with (4.5), (4.6), (1.9), (4.7) and (4.8) one gets the relation.  $\square$

Now assume that  $S$  splits into  $M$  and fix an atlas  $\mathfrak{U}$  on  $M$  which is a  $\sigma$ -splitting atlas adapted to  $(g, S)$ . Recall that if  $(U, z) \in \mathfrak{U}$  is a local chart such that  $U \cap S \neq \emptyset$  then (2.3) gives the canonical local generator  $X_{f,g}^\sigma$  of the foliation  $\mathscr{D}_{f,g}^\sigma$  (eq.  $\Xi_{f,g}^\sigma$ ) and, when  $\nu = 1$ , (2.5) gives the canonical



local generator  $X_{f,g}^{\sigma,1}$  of the foliation  $\mathcal{D}_{f,g}^{\sigma,1}$  (eq.  $\Xi_{f,g}^{\sigma,1}$ ), both associated to these coordinates. Let define  $h_0^1 = h^1(0, z^2, \dots, z^n)$  (seen as a function on  $U$ ), where  $h^1$  is the one appearing in (1.6). Since  $z^1$  is a local generator for  $\mathcal{I}_S$  there exists a germ  $k^1 \in \mathcal{O}_M$  such that

$$h^1 - h_0^1 = k^1 z^1. \quad (4.9)$$

Then we define the local holomorphic vector fields

$$\mathcal{X}_{f,g}^\sigma = k^1 z^1 \frac{\partial}{\partial z^1} + \sum_{j=2}^n h^j \frac{\partial}{\partial z^j} \quad (4.10)$$

and, only when  $\nu = 1$ ,

$$\mathcal{X}_{f,g}^{\sigma,1} = k^1 z^1 \frac{\partial}{\partial z^1} + (1 + h_0^1) \sum_{j=2}^n h^j \frac{\partial}{\partial z^j} \quad (4.11)$$

to be the *canonical local extensions* respectively of  $X_{f,g}^\sigma$  and  $X_{f,g}^{\sigma,1}$ . Observe that both generate a 1-dimensional holomorphic foliation on  $U$  leaving  $U \cap S$  invariant which restricted to  $U \cap S$  coincides respectively with  $\mathcal{D}_{f,g}^\sigma$  and  $\mathcal{D}_{f,g}^{\sigma,1}$ . Let now  $(\hat{U}, \hat{z}) \in \mathfrak{U}$  be another local chart such that  $U \cap \hat{U} \cap S \neq \emptyset$  and  $\hat{X}_{f,g}^\sigma$  and  $\hat{X}_{f,g}^{\sigma,1}$  the corresponding canonical local generators of the foliations on  $\hat{U} \cap S$  defined as in (2.3) and (2.5). Moreover let  $\hat{\mathcal{X}}_{f,g}^\sigma$  and  $\hat{\mathcal{X}}_{f,g}^{\sigma,1}$  be their canonical local extensions as defined respectively in (4.10) and (4.11). As above, there exists a germ  $a \in \mathcal{O}_M^*$  such that  $\hat{z}^1 = az^1$  and one can easily check that

$$\hat{X}_{f,g}^\sigma = \left( \frac{1}{a|_S} \right)^\nu X_{f,g}^\sigma \quad \text{and} \quad \hat{X}_{f,g}^{\sigma,1} = \frac{1}{a|_S} X_{f,g}^{\sigma,1} \quad (4.12)$$

where they overlap. We can obtain relations among their local extensions similar to the ones of Proposition 4.3 but it is not sufficient that  $S$  splits into  $M$ . In the following we use the same notation of above.

**Proposition 4.4.** *Let  $M$  be a  $n$ -dimensional complex manifold,  $S \subset M$  a non-singular connected complex hypersurface,  $(f, g) \in \text{End}_S^2(M)$  such that  $g$  is a local biholomorphism around  $S$  and set  $\nu = \nu_{f,g}$ .*

*Suppose that  $S$  is comfortably embedded into  $M$  and let  $\mathfrak{U}$  be a  $\sigma$ -comfortably atlas adapted to  $(g, S)$  (for some splitting morphism  $\sigma$ ). If  $(U, z)$  and  $(\hat{U}, \hat{z})$  are two local charts in  $\mathfrak{U}$  such that  $U \cap \hat{U} \cap S \neq \emptyset$ ,  $\mathcal{X}_{f,g}^\sigma$  and  $\hat{\mathcal{X}}_{f,g}^\sigma$  are the corresponding local holomorphic vector fields defined as in (4.10) and  $\mathcal{X}_{f,g}^{\sigma,1}$  and  $\hat{\mathcal{X}}_{f,g}^{\sigma,1}$  the ones defined as in (4.11), then for  $\nu > 1$*

$$\hat{\mathcal{X}}_{f,g}^\sigma = \left( \frac{1}{a} \right)^\nu \mathcal{X}_{f,g}^\sigma + T_1 + V_2$$

*where they overlap, with  $a \in \mathcal{O}_M^*$  such that  $\hat{z}^1 = az^1$ . Similarly, if  $\nu = 1$*

$$\hat{\mathcal{X}}_{f,g}^{\sigma,1} = \frac{1}{a} \mathcal{X}_{f,g}^{\sigma,1} + T_1 + V_2$$

where they overlap.

*Proof.* Let assume  $\nu > 1$  and focus on the first relation. Recall that since  $\mathfrak{U}$  is comfortably then

$$\left. \frac{\partial z^j}{\partial \hat{z}^1} \right|_S \equiv \left. \frac{\partial \hat{z}^j}{\partial z^1} \right|_S \equiv 0, \quad j = 2, \dots, n, \quad (4.13)$$

and  $\left. \frac{\partial^2 z^1}{(\partial \hat{z}^1)^2} \right|_S \equiv \left. \frac{\partial^2 \hat{z}^1}{(\partial z^1)^2} \right|_S \equiv 0$ , which implies

$$\left. \frac{\partial a}{\partial z^1} \right|_S \equiv \left. \frac{\partial \frac{1}{a}}{\partial \hat{z}^1} \right|_S \equiv 0. \quad (4.14)$$

Let now  $h_0^1$ ,  $k^1$  and  $\hat{h}_0^1$ ,  $\hat{k}^1$  be the functions appearing in (4.9), respectively for the coordinates  $z$  and  $\hat{z}$ . By (1.10) it follows that  $\hat{h}_0^1 a^\nu = a h_0^1 \pmod{\mathcal{I}_S}$ , since  $h^1 = h_0^1 \pmod{\mathcal{I}_S}$  and  $\hat{h}^1 = \hat{h}_0^1 \pmod{\mathcal{I}_S}$ . Moreover (4.13) and (4.14) imply that  $\left. \frac{\partial(\hat{h}_0^1 a^\nu - a h_0^1)}{\partial z^1} \right|_S \equiv 0$  hence in fact

$$\hat{h}_0^1 a^\nu = a h_0^1 \pmod{\mathcal{I}_S^2}. \quad (4.15)$$

Using (1.9), (4.14) and (4.15) one has that

$$\begin{aligned} \hat{k}^1 a^{\nu+1} z^1 &= a^\nu (\hat{h}^1 - \hat{h}_0^1) = a^\nu \hat{h}^1 - a h_0^1 \pmod{\mathcal{I}_S^2} = \\ &= a h^1 - a h_0^1 + \sum_{j=2}^n h^j \frac{\partial \hat{z}^1}{\partial z^j} \pmod{\mathcal{I}_S^{\min(\nu, 2)}} = \\ &= a k^1 z^1 + z^1 \sum_{j=2}^n h^j \frac{\partial a}{\partial z^j} \pmod{\mathcal{I}_S^{\min(\nu, 2)}}, \end{aligned}$$

then since  $\nu > 1$  one gets the relation

$$\hat{k}^1 a^{\nu+1} = a k^1 + \sum_{j=2}^n h^j \frac{\partial a}{\partial z^j} \pmod{\mathcal{I}_S}, \quad (4.16)$$

which in practice plays here the role of (4.7). Then manipulating

$$\hat{\mathcal{X}}_{f,g}^\sigma = \hat{k}^1 \hat{z}^1 \frac{\partial}{\partial \hat{z}^1} + \sum_{j=2}^n \hat{h}^j \frac{\partial}{\partial \hat{z}^j}$$

with the previous (4.5), (4.6), (1.9), (4.8) and with (4.13), (4.14), (4.16), and recalling that  $\nu > 1$ , one gets the first relation.

Let now assume  $\nu = 1$  and focus on the second relation. Observe that (4.16) is no longer true in this case but we need something similar. Since

$$\begin{aligned} a \hat{h}^1 z^1 &= \hat{h}^1 \hat{z}^1 = \hat{f}^1 - \hat{g}^1 = \\ &= z^1 \sum_{j=1}^n h^j \frac{\partial \hat{z}^1}{\partial z^j} + \frac{1}{2} (z^1)^2 \sum_{j,k=1}^n h^j h^k \frac{\partial^2 \hat{z}^1}{\partial z^j \partial z^k} \pmod{\mathcal{I}_S^3} \end{aligned}$$

then one can easily gets

$$a\hat{h}^1 = ah^1 + (1 + h_0^1) z^1 \sum_{j=2}^n h^j \frac{\partial a}{\partial z^j} \pmod{\mathcal{I}_S^2}.$$

From this and using (4.15) again (with  $\nu = 1$ ), one obtains

$$a^2 \hat{k}^1 = ak^1 + (1 + h_0^1) \sum_{j=2}^n h^j \frac{\partial a}{\partial z^j} \pmod{\mathcal{I}_S}, \quad (4.17)$$

then manipulating

$$\hat{\mathcal{X}}_{f,g}^{\sigma,1} = \hat{k}^1 \hat{z}^1 \frac{\partial}{\partial \hat{z}^1} + (1 + \hat{h}_0^1) \sum_{j=2}^n \hat{h}^j \frac{\partial}{\partial \hat{z}^j}$$

with the previous (4.5), (4.6), (1.9), (4.8), (4.13), (4.14) and with (4.17) one concludes.  $\square$

**Remark 4.5.** If we consider the couple  $(f, \text{Id}_M)$  then Proposition 4.3 turns out to be [3, Lem.4.1.], while Proposition 4.4 turns out to be [3, Prop.4.2.].

## 5. A CAMACHO-SAD-TYPE INDEX THEOREM

Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a globally irreducible complex hypersurface whose regular part is  $S' = S - \text{Sing}(S)$ . Let  $(f, g) \in \text{End}_S^2(M)$  be a couple such that  $g$  is a local biholomorphism around  $S'$  and with order of coincidence  $\nu = \nu_{f,g}$ .

Let use the following notation in the sequel. When  $(f, g)$  is tangential along  $S$  set  $\mathcal{D} = \mathcal{D}_{f,g}$ ,  $\Xi = \Xi_{f,g}$  and let  $\mathfrak{U}$  denote an atlas adapted to  $(g, S')$ . If  $(U, z) \in \mathfrak{U}$  is a local chart then set also  $X = X_{f,g}$  and  $\mathcal{X} = \mathcal{X}_{f,g}$  as defined in (2.1) and (4.3). Instead, when  $S'$  is comfortably embedded into  $M$  set  $\mathcal{D} = \mathcal{D}_{f,g}^\sigma$  and  $\Xi = \Xi_{f,g}^\sigma$  if  $\nu > 1$  while set  $\mathcal{D} = \mathcal{D}_{f,g}^{\sigma,1}$  and  $\Xi = \Xi_{f,g}^{\sigma,1}$  if  $\nu = 1$ . Moreover let  $\mathfrak{U}$  denote a  $\sigma$ -comfortably atlas adapted to  $(g, S')$  (for some splitting morphism  $\sigma$ ) and if  $(U, z) \in \mathfrak{U}$  is a local chart then set  $X = X_{f,g}^\sigma$  and  $\mathcal{X} = \mathcal{X}_{f,g}^\sigma$  as defined in (2.3) and (4.10) when  $\nu > 1$ , while set  $X = X_{f,g}^{\sigma,1}$  and  $\mathcal{X} = \mathcal{X}_{f,g}^{\sigma,1}$  as defined in (2.5) and (4.11) when  $\nu = 1$ . In all the cases let denote  $S^0 = S' - \text{Sing}(\mathcal{D})$  and  $\mathcal{D}^0 = \mathcal{D}|_{S^0}$ .

Let  $(U, z) \in \mathfrak{U}$  be such that  $U \cap S^0 \neq \emptyset$ . As already observed the local holomorphic vector field  $\mathcal{X}$  generates a 1-dimensional holomorphic foliation on  $U$  which extends  $\mathcal{D}|_{U \cap S'}$ , then locally we are in the situation described at the beginning of Section 4. This means that for any  $(U, z) \in \mathfrak{U}$  (such that  $U \cap S^0 \neq \emptyset$ ) we can define a partial holomorphic connection on  $N_{U \cap S^0}$  along  $N_{U \cap S^0}^{\otimes \nu} \equiv \Xi|_{U \cap S^0} \subset TS'|_{U \cap S^0}$  as in (4.1), that is

$$\begin{aligned} \mathcal{N}_{U \cap S^0} &\xrightarrow{\delta_U^{cs}} (\mathcal{N}_{U \cap S^0}^{\otimes \nu})^* \otimes \mathcal{N}_{U \cap S^0} \\ w &\longrightarrow \delta_U^{cs}(w) \text{ s.t. } \delta_U^{cs}(w)(\varphi X) = \varphi \pi([\mathcal{X}, \tilde{w}]|_{S^0}) \end{aligned} \quad (5.1)$$

for any  $w \in \mathcal{N}_{U \cap S^0}$  and  $\varphi \in \mathcal{O}_{U \cap S^0}$ , where  $\pi : TM|_{S^0} \rightarrow N_{S^0}$  is the projection and  $\tilde{w} \in \mathcal{T}_M|_{U \cap S^0}$  is any vector field such that  $\pi(\tilde{w}|_{U \cap S^0}) = w$ . Observe that since  $X$  is a local generator of  $\mathcal{D}$  on  $U \cap S'$  then any local vector field  $v \in \mathcal{N}_{U \cap S^0}^{\otimes \nu}$  is of the form  $v = \varphi X$  for some function  $\varphi \in \mathcal{O}_{U \cap S^0}$ .

The key point is that we can glue together all these local partial holomorphic connections by Proposition 4.3 and 4.4 and consequently define a (global) partial holomorphic connection on  $N_{S^0}$  along  $N_{S^0}^{\otimes \nu} \equiv \Xi|_{S^0} \subset TS^0$ .

**Proposizione 5.1.** *Let  $M$  be a  $n$ -dimensional complex manifold,  $S \subset M$  a globally irreducible complex hypersurface and  $(f, g) \in \text{End}_S^2(M)$  a couple such that  $g$  is a local biholomorphism around  $S'$  with order of coincidence  $\nu = \nu_{f,g}$ . Assume that  $(f, g)$  is tangential along  $S$  or that  $S'$  is comfortably embedded into  $M$  and use the notation introduced above.*

*Then if  $(U, z)$  and  $(\hat{U}, \hat{z})$  are two local charts in  $\mathfrak{A}$  such that  $U \cap \hat{U} \cap S^0 \neq \emptyset$  and  $\delta_U^{cs}$  and  $\delta_{\hat{U}}^{cs}$  are the corresponding local partial holomorphic connections defined as in (5.1) we have that*

$$\delta_U^{cs} = \delta_{\hat{U}}^{cs}$$

*where they overlap. Consequently, there is a well-defined partial holomorphic connection*

$$\mathcal{N}_{S^0} \xrightarrow{\delta_{\mathcal{D}}^{cs}} (\mathcal{N}_{S^0}^{\otimes \nu})^* \otimes \mathcal{N}_{S^0}$$

*on  $N_{S^0}$  along  $N_{S^0}^{\otimes \nu}$ .*

*Proof.* Since  $X$  is a local generator of  $\mathcal{D}$  on  $U \cap \hat{U} \cap S^0$  we just need to prove that

$$\delta_{\hat{U}}^{cs}(w)(X) = \delta_U^{cs}(w)(X)$$

for any  $w \in \mathcal{N}_{U \cap \hat{U} \cap S^0}$ . This is true because if  $a \in \mathcal{O}_M^*$  is such that  $\hat{z}^1 = az^1$  and  $\pi : TM|_{S^0} \rightarrow N_{S^0}$  is the projection then by (4.4), (4.12) and by Proposition 4.3 and 4.4 it follows that

$$\begin{aligned} \delta_{\hat{U}}^{cs}(w)(X) &= (a|_{S^0})^\nu \delta_{\hat{U}}^{cs}(w)(\hat{X}) = (a|_{S^0})^\nu \pi \left( \left[ \hat{\mathcal{X}}, \tilde{w} \right] \Big|_{S^0} \right) = \\ &= (a|_{S^0})^\nu \pi \left( \left[ \left( \frac{1}{a} \right)^\nu \mathcal{X} + T_1 + V_2, \tilde{w} \right] \Big|_{S^0} \right) = \\ &= \pi \left( [\mathcal{X}, \tilde{w}] \Big|_{S^0} \right) = \delta_U^{cs}(w)(X), \end{aligned}$$

where the last but one equality is due to the fact that  $[T_1, \tilde{w}]|_{S^0} \in \mathcal{T}_{S^0}$  and  $[V_2, \tilde{w}]|_{S^0} \equiv 0$ .  $\square$

Now recall that a hypersurface  $S \subset M$  defines naturally a line bundle on  $M$ , usually denoted by  $[S]$ , and that  $[S]|_{S'} \cong N_{S'}$ . Moreover  $S$  can be seen as the zero locus of a holomorphic section of  $[S]$ . Then by Proposition 5.1 and by [20, Th.IV.2.4.] (or [11, Th.5.3.7.]) we have the following theorem.

**Theorem 5.2** (Camacho-Sad-type index theorem). *Let  $M$  be a  $n$ -dimensional complex manifold,  $S \subset M$  a globally irreducible compact complex hypersurface whose regular part is  $S' = S - \text{Sing}(S)$  and  $(f, g) \in \text{End}_S^2(M)$*

a couple such that  $g$  is a local biholomorphism around  $S'$  with order of coincidence  $\nu = \nu_{f,g}$ . Assume that

(i)  $(f, g)$  is tangential along  $S$

or that

(ii)  $S'$  is comfortably embedded into  $M$ .

In case (i) set  $\mathcal{D} = \mathcal{D}_{f,g}$  while in case (ii) set  $\mathcal{D} = \mathcal{D}_{f,g}^\sigma$  when  $\nu > 1$  or  $\mathcal{D} = \mathcal{D}_{f,g}^{\sigma,1}$  when  $\nu = 1$  (where  $\sigma$  is some splitting morphism) and suppose  $\mathcal{D} \neq \emptyset$ . Let  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D}) = \sqcup_\lambda \Sigma_\lambda$  be the decomposition in connected components of the singular set  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D})$ .

Then there exist complex numbers  $\text{Res}(\mathcal{D}; S; \Sigma_\lambda)$  such that

$$\sum_\lambda \text{Res}(\mathcal{D}; S; \Sigma_\lambda) = \int_S c_1^{n-1}([S]),$$

where  $c_1([S])$  denotes the first Chern class of the line bundle  $[S]$ .

We call Theorem 5.2 a *Camacho-Sad-type index theorem* because it is basically inspired by the Camacho-Sad index theorem proved in [12, Appendix]. To be precise if  $n = 2$ ,  $S$  is a non-singular curve and  $g = \text{Id}_M$  then Theorem 5.2 is broadly the residual index theorem [1, Th.1.1.], which was inspired by the Camacho-Sad index theorem just mentioned. See the survey [2] for a clear exposition, without too many details, of a general procedure to obtain Camacho-Sad-type index theorems. A full explanation of this procedure and some resulting theorems may be found in [4].

**Remark 5.3.** If we consider the couple  $(f, \text{Id}_M)$  then Theorem 5.2 turns out to be [3, Th.6.2.], which itself generalize [1, Th.1.1.]. Observe that in their notation  $\text{Sing}(f, \text{Id}_M)$  is  $\text{Sing}(f)$  and they assume  $S' - \text{Sing}(f)$  to be comfortably embedded in  $M$ , not  $S'$ . In fact their assumption is sufficient.

We conclude this section by deriving explicit formulas for the computation of the residues in Theorem 5.2 at isolated singular points. First, we briefly recall how these residues are defined in Lehmann-Suwa theory. As a reference see [20, Sec.IV.2. and IV.6.].

Let  $U^0 \subset M$  be a tubular neighborhood of  $S^0$  (it always exists, see for example [18, p.465]) and  $r : U^0 \rightarrow S^0$  the associated smooth retraction. Observe that  $r^*N_{S^0} \cong [S]|_{U^0}$  hence if  $\nabla_{\mathcal{D}}^{cs}$  is any  $(1, 0)$ -type extension of  $\delta_{\mathcal{D}}^{cs}$  then  $r^*\nabla_{\mathcal{D}}^{cs}$  is a smooth connection on  $[S]|_{U^0}$  such that, by the Bott vanishing theorem,

$$c_1^{n-1}(r^*\nabla_{\mathcal{D}}^{cs}) = r^*c_1^{n-1}(\nabla_{\mathcal{D}}^{cs}) = 0.$$

Let  $\Sigma$  be a connected component of  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D})$ ,  $U \subset M$  any open neighborhood of it such that  $U \cap (\text{Sing}(S) \cup \text{Sing}(\mathcal{D})) = \Sigma$  and take any connection  $\nabla_U$  on  $[S]|_U$ . Lastly, let  $\tilde{R} \subset U$  be any compact real sub-manifold of dimension  $2n$  oriented as  $M$  and such that  $\Sigma \subset \text{int}(\tilde{R})$ . Assume its boundary  $\partial\tilde{R}$  transverse to  $S$  and with the orientation induced by  $\tilde{R}$ . Set

$R = \tilde{R} \cap S$  and  $\partial R = \partial \tilde{R} \cap S$ . Then by definition the residue is

$$\text{Res}(\mathcal{D}; S; \Sigma) = \int_R c_1^{n-1}(\nabla_U) - \int_{\partial R} c_1^{n-1}(r^* \nabla_{\mathcal{D}}^{cs}, \nabla_U),$$

where  $c_1^{n-1}(r^* \nabla_{\mathcal{D}}^{cs}, \nabla_U)$  is the Bott difference form. One can show that this formula does not depend on the various choices done.

In particular, if  $\Sigma = \{p\}$  we can suppose  $U$  to be such that  $[S]|_U$  is trivial and then take  $\nabla_U$  trivial respect to some generator. With this choice

$$\text{Res}(\mathcal{D}; S; p) = - \int_{\partial R} c_1^{n-1}(r^* \nabla_{\mathcal{D}}^{cs}, \nabla_U)$$

and since  $\partial R \subset U \cap S^0$  and  $(r^* \nabla_{\mathcal{D}}^{cs})|_{S^0} = \nabla_{\mathcal{D}}^{cs}$  we can in fact rewrite it as

$$\text{Res}(\mathcal{D}; S; p) = - \int_{\partial R} c_1^{n-1}(\nabla_{\mathcal{D}}^{cs}, \nabla_U|_{U \cap S^0}).$$

Observe that  $\nabla_U|_{U \cap S^0}$  is a smooth connection on  $N_{U \cap S^0}$ .

Assume now that  $p \in \text{Sing}(\mathcal{D}) - \text{Sing}(S)$ . In this case  $N_{U \cap S}$  does exist and we can substitute  $\nabla_U|_{U \cap S^0}$  with a smooth connection  $\nabla_{U \cap S}$  on  $N_{U \cap S}$  trivial respect to some local generator of it, thus

$$\text{Res}(\mathcal{D}; S; p) = - \int_{\partial R} c_1^{n-1}(\nabla_{\mathcal{D}}^{cs}, \nabla_{U \cap S}).$$

At this point one can argue as in the proof of [20, Th.III.5.5.] and obtain explicit formulas. In particular, let  $(U, z)$  be a local chart of  $M$  at  $p$  belonging to  $\mathfrak{U}$  and the  $h^j \in \mathcal{O}_{M,p}$  as in (1.6). Choose as  $\nabla_{U \cap S}$  the connection trivial respect to  $\partial z^1$ . Then when  $(f, g)$  is tangential along  $S$  and  $\mathcal{D} = \mathcal{D}_{f,g}$  we get the formula

$$\text{Res}(\mathcal{D}; S; p) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_{\Gamma} \frac{(\ell^1)^{n-1}}{h^2 \dots h^n} dz^2 \wedge \dots \wedge dz^n, \quad (5.2)$$

where  $\ell^1 \in \mathcal{O}_{M,p}$  is the one such that  $h^1 = \ell^1 z^1$  and  $\Gamma$  is the  $(n-1)$  cycle

$$\Gamma = \{q \in U \cap S \text{ s.t. } |h^2(q)| = \dots = |h^n(q)| = \epsilon\},$$

for  $\epsilon > 0$  small enough, oriented so that  $d\vartheta^2 \wedge \dots \wedge d\vartheta^n > 0$  where  $\vartheta^j = \arg(h^j)$  for  $j = 2, \dots, n$ . When  $S'$  is comfortably embedded into  $M$ ,  $\nu > 1$  and  $\mathcal{D} = \mathcal{D}_{f,g}^\sigma$  we get instead the formula

$$\text{Res}(\mathcal{D}; S; p) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_{\Gamma} \frac{(k^1)^{n-1}}{h^2 \dots h^n} dz^2 \wedge \dots \wedge dz^n, \quad (5.3)$$

where  $k^1$  is the one appearing in (4.9) and  $\Gamma$  is as above. Finally, when  $S'$  is comfortably embedded into  $M$ ,  $\nu = 1$  and  $\mathcal{D} = \mathcal{D}_{f,g}^{\sigma,1}$  we have

$$\text{Res}(\mathcal{D}; S; p) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_{\Gamma'} \frac{(k^1)^{n-1}}{(1 + h_0^1)^{n-1} h^2 \dots h^n} dz^2 \wedge \dots \wedge dz^n, \quad (5.4)$$

where  $h_0^1$  is the one appearing in (4.9) and

$$\Gamma' = \{q \in U \cap S \text{ s.t. } |(1 + h^1(q))h^2(q)| = \cdots = |(1 + h^1(q))h^n(q)| = \epsilon\}.$$

**Remark 5.4.** Let  $p \in \text{Sing}(\mathcal{D}) - \text{Sing}(S)$  and  $(U, z) \in \mathfrak{U}$  be a local chart at  $p$ . By construction  $\delta_{\mathcal{D}}^{cs}$  is locally induced by the natural ‘holomorphic action’ of  $X$  on  $N_{U \cap S^0}$  by Lie bracket then we could have used directly the formula just after [16, Th.2] (which is the same of [20, Th.IV.6.3.]) to get (5.2), (5.3) and (5.4). See also [20, Rmk.IV.6.7.(1) and (4)].

We end treating the case  $p \in \text{Sing}(S)$ , which is more complicated since  $N_{U \cap S}$  does not exist and in general there is no natural local extension of  $\mathcal{D}$  at such a point. Anyway, assuming that  $g$  is a biholomorphism in a neighborhood of  $p$  we can calculate explicitly the residue when  $n = 2$  and, in some cases, when  $n > 2$ .

Since  $p \in \text{Sing}(S)$  we do not have a local chart  $(U, z) \in \mathfrak{U}$  at  $p$  but when  $n = 2$  we can ‘almost’ find it. In fact, let  $U \subset M$  be an open neighborhood of  $p$  such that  $U \cap (\text{Sing}(\mathcal{D}) \cup \text{Sing}(S)) = \{p\}$  and  $g|_U$  is a biholomorphism onto its image. We can assume to have on it a local generator  $y$  of  $\mathcal{I}_S$  and some coordinates  $(u^1, u^2)$  such that  $dy \wedge du^2 \neq 0$  on  $U \cap S^0$ . In particular  $U \cap S = \{y = 0\}$  and we can suppose (possibly shrinking  $U$ ) that  $(U - \{p\}, (y, u^2))$  is a local chart in  $\mathfrak{U}$ . Since  $y$  generates  $\mathcal{I}_S$  on  $U$ , the dual of  $[y] \in \mathcal{I}_S/\mathcal{I}_S^2$  defines a local generator of  $[S]$  on  $U$  whose restriction to  $U \cap S^0$  coincides with the local generator  $\partial y = \pi(\frac{\partial}{\partial y}|_{U \cap S^0})$  of  $N_{U \cap S^0}$ . We can then choose  $\nabla_U$  to be trivial respect to the dual of  $[y]$  and consequently  $\nabla_U|_{U \cap S^0}$  is trivial respect to  $\partial y$ . With this choice we can again argue as in the proof of [20, Th.III.5.5.] and obtain explicit formulas likewise the previous ones. Referring to the notation of (5.2), (5.3) and (5.4) and reminding how are defined the special coordinates  $(w^1, w^2)$  associated to the  $(z^1, z^2)$  observe that when  $(f, g)$  is tangential along  $S$

$$\frac{\ell^1}{h^2} \Big|_{S^0} = \frac{f^1 - g^1}{z^1(f^2 - g^2)} \Big|_{S^0} = \frac{z^1 \circ g^{-1} \circ f - z^1}{z^1(z^2 \circ g^{-1} \circ f - z^2)} \Big|_{S^0}.$$

Instead when  $S'$  is comfortably embedded into  $M$ , if  $\nu > 1$

$$\frac{k^1}{h^2} \Big|_{S^0} = \frac{f^1 - g^1 - (z^1)^\nu h_0^1}{z^1(f^2 - g^2)} \Big|_{S^0} = \frac{z^1 \circ g^{-1} \circ f - z^1 - (z^1)^\nu h_0^1}{z^1(z^2 \circ g^{-1} \circ f - z^2)} \Big|_{S^0},$$

while if  $\nu = 1$

$$\begin{aligned} \frac{k^1}{(1 + h_0^1)h^2} \Big|_{S^0} &= \frac{f^1 - g^1 - z^1 h_0^1}{(z^1 + z^1 h_0^1)(f^2 - g^2)} \Big|_{S^0} = \\ &= \frac{z^1 \circ g^{-1} \circ f - z^1 - z^1 h_0^1}{(z^1 \circ g^{-1} \circ f)(z^2 \circ g^{-1} \circ f - z^2)} \Big|_{S^0}, \end{aligned}$$

where we use the fact that  $h_0^1|_{S^0} = h^1|_{S^0}$ . Hence substituting the coordinates  $(z^1, z^2)$  with the  $(y, u^2)$  and using these equalities we have the following.

When  $(f, g)$  is tangential along  $S$  and  $\mathcal{D} = \mathcal{D}_{f,g}$  by (5.2) we get

$$\text{Res}(\mathcal{D}; S; p) = \frac{1}{2\pi i} \int_{\gamma} \frac{y \circ g^{-1} \circ f - y}{y(u^2 \circ g^{-1} \circ f - u^2)} du^2, \quad (5.5)$$

where  $\gamma \subset U \cap S^\circ$  is any simple closed curve around  $p$  with the orientation induced by the one of  $S'$ . When  $S'$  is comfortably embedded into  $M$ ,  $\nu > 1$  and  $\mathcal{D} = \mathcal{D}_{f,g}^\sigma$  by (5.3) we get instead

$$\text{Res}(\mathcal{D}; S; p) = \frac{1}{2\pi i} \int_{\gamma} \frac{y \circ g^{-1} \circ f - y - y^\nu b}{y(u^2 \circ g^{-1} \circ f - u^2)} du^2, \quad (5.6)$$

where  $b = \frac{y \circ g^{-1} \circ f - y}{y^\nu}$  and  $\gamma$  as before. Finally, if  $S'$  is comfortably embedded into  $M$ ,  $\nu = 1$  and  $\mathcal{D} = \mathcal{D}_{f,g}^{\sigma,1}$  by (5.4) we obtain

$$\text{Res}(\mathcal{D}; S; p) = \frac{1}{2\pi i} \int_{\gamma} \frac{y \circ g^{-1} \circ f - y - yb}{y \circ g^{-1} \circ f (u^2 \circ g^{-1} \circ f - u^2)} du^2, \quad (5.7)$$

with  $b$  and  $\gamma$  as before.

If  $n > 2$  we are able to compute explicitly the residue at  $p \in \text{Sing}(S)$  when  $(f, g)$  is tangential along  $S$  and  $\nu > 1$ . In fact, with these hypotheses we can define a local holomorphic vector field around  $p$  leaving  $S$  invariant with (possibly) an isolated singularity at  $p$ , whose natural ‘holomorphic action’ produced by Lie bracket on  $N_{S^0}$  (see [20, Sec.IV.6.]) induces locally  $\delta_{\mathcal{D}}^{cs}$ . In this way we can apply directly the formula just after [16, Th.2].

For this purpose, let  $U \subset M$  be an open neighborhood of  $p$  such that  $U \cap (\text{Sing}(S) \cup \text{Sing}(sD)) = \{p\}$  and  $g|_U$  is a biholomorphism onto its image. Suppose to have a local generator  $y$  of  $\mathcal{I}_S$  and any coordinates  $(u^1, \dots, u^n)$  on it, with  $(v^1 = u^1 \circ g^{-1}, \dots, v^n = u^n \circ g^{-1})$  the corresponding ‘special’ coordinates on  $g(U)$  at  $f(p)$ . Let define the holomorphic vector field on  $U$

$$\mathcal{V}_{f,g} = \sum_{j=1}^n \frac{v^j \circ f - v^j \circ g}{y^\nu} \frac{\partial}{\partial u^j} \quad (5.8)$$

and  $U^j = \{x \in U \text{ s.t. } \frac{\partial y}{\partial u^j}(x) \neq 0\}$  for  $j = 1, \dots, n$ . Observe that  $\sqcup_{j=1}^n U^j$  is an open subset of  $U - \{p\}$  containing  $U \cap S^0$  and in particular the sets  $U^j \cap S^0$  cover  $U \cap S^0$ . We can put on each  $U^j$  the coordinates  $z_j = (z_j^1, \dots, z_j^n)$  adapted to  $(g, S')$  defined by

$$\begin{aligned} z_j^1 &= y, \\ z_j^i &= u^{i-1}, & \text{for } i = 2, \dots, j, \\ z_j^i &= u^i, & \text{for } i = j+1, \dots, n. \end{aligned}$$

Clearly  $(U^j, z_j) \in \mathfrak{U}$  for every  $j = 1, \dots, n$  and let  $w_j = (w_j^1, \dots, w_j^n)$  be the usual special coordinates associated to the  $z_j$ . If  $\mathcal{X}^j$  is the local holomorphic vector field associated to the chart  $(U^j, z_j)$  defined as in (4.3) then one can prove that

$$\mathcal{V}_{f,g}|_{U^j} = \mathcal{X}^j + V_\nu^j, \quad \text{for } j = 1, \dots, n, \quad (5.9)$$



where  $V_\nu^j$  is a holomorphic vector field on  $U^j$  whose coefficients are in  $\mathcal{I}_S^\nu$ . We show (5.9) only for  $j = 1$  since the other cases can be proved in the same way. For simplicity let denote the coordinates by  $z = (z^1, \dots, z^n)$  and  $w = (w^1, \dots, w^n)$ . Let  $h^1, \dots, h^n$  be the corresponding germs as in (1.6), then on  $U^1$

$$\frac{v^i \circ f - v^i \circ g}{y^\nu} = \frac{w^i \circ f - w^i \circ g}{(z^1)^\nu} = h^i$$

for  $i = 2, \dots, n$ , while reminding (1.1)

$$v^1 \circ f - v^1 \circ g = \left( \frac{\partial y}{\partial u^1} \right)^{-1} h^1 y^\nu - \left( \frac{\partial y}{\partial u^1} \right)^{-1} \sum_{k=2}^n \frac{\partial y}{\partial u^k} h^k y^\nu \pmod{\mathcal{I}_S^{2\nu}}$$

hence

$$\frac{v^1 \circ f - v^1 \circ g}{y^\nu} = \left( \frac{\partial y}{\partial u^1} \right)^{-1} \left[ h^1 - \sum_{k=2}^n \frac{\partial y}{\partial u^k} h^k \right] \pmod{\mathcal{I}_S^\nu}.$$

Therefore putting these equalities into (5.8) and noting that  $\frac{\partial}{\partial u^1} = \frac{\partial y}{\partial u^1} \frac{\partial}{\partial z^1}$  and  $\frac{\partial}{\partial u^i} = \frac{\partial y}{\partial u^i} \frac{\partial}{\partial z^1} + \frac{\partial}{\partial z^i}$  for  $i = 2, \dots, n$  we done.

By (5.9) it follows that  $\mathcal{V}_{f,g}$  generates a 1-dimensional holomorphic foliation on  $U$  leaving  $U \cap S^0$  invariant, which restricted to  $U \cap S^0$  coincides with  $\mathcal{D}_{f,g}$  and having an isolated singularity at  $p$ . Moreover taking into account that  $\nu > 1$  and reminding (5.1), equation (5.9) shows even that the natural ‘holomorphic action’ of  $\mathcal{V}_{f,g}$  on  $N_{U \cap S^0}$  induces locally  $\delta_{\mathcal{D}}^{\text{cs}}$ . Hence choosing the coordinates  $(u^1, \dots, u^n)$  in such a way that  $\{y = (v^2 \circ f - v^2 \circ g)/y^\nu = \dots = (v^n \circ f - v^n \circ g)/y^\nu = 0\} = \{p\}$  (we can do that, see for example [20, Cor.IV.4.5.]) then we can express the residue by the formula just after [16, Th.2] (eq. [20, Th.IV.6.3.]). In particular, observing that

$$\mathcal{V}_{f,g}(y) = \left( \sum_{j=1}^n \frac{v^j \circ f - v^j \circ g}{y^{\nu+1}} \frac{\partial y}{\partial u^j} \right) y$$

we get the formula

$$\text{Res}(\mathcal{D}; S; p) = \left( \frac{-i}{2\pi} \right)^{n-1} \int_{\Gamma} \frac{\left[ \sum_{j=1}^n (v^j \circ f - v^j \circ g) \frac{\partial y}{\partial u^j} \right]^{n-1}}{y^{n-1} \prod_{j=2}^n (v^j \circ f - v^j \circ g)} du^2 \wedge \dots \wedge du^n, \quad (5.10)$$

where  $\Gamma$  is the  $(n-1)$  cycle

$$\Gamma = \left\{ q \in U \cap S \text{ s.t. } \left| \frac{v^j \circ f - v^j \circ g}{y^\nu}(q) \right| = \epsilon, \text{ for } j = 2, \dots, n \right\},$$

for  $\epsilon > 0$  small enough, oriented as usual.

**Remark 5.5.** If  $n = 2$ ,  $(f, g)$  is tangential along  $S$  and  $\nu > 1$  we can argue as just done, taking as local chart  $(U, u)$  at  $p$  one such that  $dy \wedge du^2 \neq 0$  on  $U \cap S^0$ . Thus we can assume (possibly shrinking  $U$ ) that  $U - \{p\} = U^1$

and then that  $(U - \{p\}, (y, u^2)) \in \mathfrak{U}$ . If  $\mathcal{X}$  is the local holomorphic vector field associated to  $(y, u^2)$  as in (4.3) (defined on  $U - \{p\}$ ) then by (5.9) the natural ‘holomorphic actions’ of  $\mathcal{V}_{f,g}$  and  $\mathcal{X}$  on  $N_{U \cap S^0}$  are in fact the same. Then suitably modifying (5.10) we recover (5.5).

Lastly, observe that if  $p$  was a non-singular point for  $S$  then we could take a local chart  $(U, u) = (U, z) \in \mathfrak{U}$  and as local generator of  $\mathcal{I}_S$  the function  $y = z^1$ . With these choices clearly  $\mathcal{V}_{f,g} = \mathcal{X}$  and then (5.10) would be (5.2).

## 6. A LEHMANN-SUWA-TYPE INDEX THEOREM

Let  $M$ ,  $S$  and  $(f, g)$  be as in Section 5 and set as usual  $\nu = \nu_{f,g}$  and  $S' = S - \text{Sing}(S)$ . From now on assume  $(f, g)$  tangential along  $S$  and let denote  $\mathcal{D} = \mathcal{D}_{f,g}$ ,  $S^0 = S' - \text{Sing}(\mathcal{D})$  and  $\Xi = \Xi_{f,g}$  for simplicity. Moreover let  $\mathfrak{U}$  be an atlas adapted to  $(g, S')$  and if  $(U, z) \in \mathfrak{U}$  set also  $X = X_{f,g}$  and  $\mathcal{X} = \mathcal{X}_{f,g}$ , as defined respectively in (2.1) and (4.3).

Let  $(U, z) \in \mathfrak{U}$  be such that  $U \cap S^0 \neq \emptyset$ . As in Section 5 the local holomorphic vector field  $\mathcal{X}$  generates a 1-dimensional holomorphic foliation on  $U$  which extends  $\mathcal{D}|_{U \cap S'}$ , hence we can define a partial holomorphic connection on  $N_{\mathcal{D}}^M|_{U \cap S^0}$  along  $N_{U \cap S^0}^{\otimes \nu} \equiv \Xi|_{U \cap S^0} \subset TS'|_{U \cap S^0}$  as in (4.2), that is

$$\begin{aligned} \mathcal{O}(N_{\mathcal{D}}^M|_{U \cap S^0}) &\xrightarrow{\delta_U^{ls}} (\mathcal{N}_{U \cap S^0}^{\otimes \nu})^* \otimes \mathcal{O}(N_{\mathcal{D}}^M|_{U \cap S^0}) \\ w &\longrightarrow \delta_U^{ls}(w) \text{ s.t. } \delta_U^{ls}(w)(\varphi X) = \varphi \rho([\mathcal{X}, \tilde{w}]|_{S^0}) \end{aligned} \quad (6.1)$$

for any  $w \in \mathcal{O}(N_{\mathcal{D}}^M|_{U \cap S^0})$  and  $\varphi \in \mathcal{O}_{U \cap S^0}$ , where  $\rho : TM|_{S^0} \rightarrow N_{\mathcal{D}}^M$  is the projection and  $\tilde{w} \in \mathcal{T}_M|_{U \cap S^0}$  is any vector field such that  $\rho(\tilde{w}|_{U \cap S^0}) = w$ . Observe that since  $X$  is a local generator of  $\mathcal{D}$  on  $U \cap S'$  then any local vector field  $v \in \mathcal{N}_{U \cap S^0}^{\otimes \nu}$  is of the form  $v = \varphi X$  for some  $\varphi \in \mathcal{O}_{U \cap S^0}$ .

By Proposition 4.3 we can glue together all these local partial holomorphic connections when  $\nu > 1$  and then define a (global) partial holomorphic connection on  $N_{\mathcal{D}}^M$  along  $N_{S^0}^{\otimes \nu} \equiv \Xi|_{S^0} \subset TS^0$ .

**Proposizione 6.1.** *Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a globally irreducible complex hypersurface. Let  $(f, g) \in \text{End}_S^2(M)$  be a couple such that  $g$  is a local biholomorphism around  $S'$ , with order of coincidence  $\nu = \nu_{f,g} > 1$  and tangential along  $S$ .*

*Then, referring to the notation introduced above, if  $(U, z)$  and  $(\hat{U}, \hat{z})$  are two local charts in  $\mathfrak{U}$  such that  $U \cap \hat{U} \cap S^0 \neq \emptyset$  and  $\delta_U^{ls}$  and  $\delta_{\hat{U}}^{ls}$  are the corresponding local partial holomorphic connections defined as in (6.1) we have that*

$$\delta_U^{ls} = \delta_{\hat{U}}^{ls}$$

*where they overlap. Consequently, there is a well-defined partial holomorphic connection*

$$\mathcal{O}(N_{\mathcal{D}}^M) \xrightarrow{\delta_S^{ls}} (\mathcal{N}_{S^0}^{\otimes \nu})^* \otimes \mathcal{O}(N_{\mathcal{D}}^M)$$

*on  $N_{\mathcal{D}}^M$  along  $N_{S^0}^{\otimes \nu}$ .*

*Proof.* Since  $X$  is a local generator of  $\mathcal{D}$  on  $U \cap \hat{U} \cap S^0$  we just need to prove that

$$\delta_{\hat{U}}^{ls}(w)(X) = \delta_U^{ls}(w)(X)$$

for any  $w \in \mathcal{O}(N_{\mathcal{D}}^M|_{U \cap \hat{U} \cap S^0})$ . This is true because if  $a \in \mathcal{O}_M^*$  is such that  $\hat{z}^1 = az^1$  and  $\rho : TM|_{S^0} \rightarrow N_{\mathcal{D}}^M$  is the projection then by (4.4) and Proposition 4.3, recalling that  $\nu > 1$ , it follows

$$\begin{aligned} \delta_{\hat{U}}^{ls}(w)(X) &= (a|_{S^0})^\nu \delta_{\hat{U}}^{ls}(w)(\hat{X}) = (a|_{S^0})^\nu \rho \left( \left[ \hat{\mathcal{X}}, \tilde{w} \right] \Big|_{S^0} \right) = \\ &= (a|_{S^0})^\nu \rho \left( \left[ \left( \frac{1}{a} \right)^\nu \mathcal{X} + V_2, \tilde{w} \right] \Big|_{S^0} \right) = \\ &= \rho([\mathcal{X}, \tilde{w}]|_{S^0}) = \delta_U^{ls}(w)(X), \end{aligned}$$

where the last but one equality is due to the fact that  $[V_2, \tilde{w}]|_{S^0} \equiv 0$ .  $\square$

Recall again that a hypersurface  $S \subset M$  defines naturally a line bundle  $[S]$  on  $M$  such that  $[S]|_{S'} \cong N_{S'}$ . Then by Proposition 6.1 and arguing as at [20, p.130] we get the following theorem.

**Theorem 6.2** (Lehmann-Suwa-type index theorem). *Let  $M$  be a  $n$ -dimensional complex manifold and  $S \subset M$  a globally irreducible compact complex hypersurface. Let  $(f, g) \in \text{End}_S^2(M)$  be a couple such that  $g$  is a local biholomorphism around  $S'$ , with order of coincidence  $\nu = \nu_{f,g} > 1$  and tangential along  $S$ . Set  $\mathcal{D} = \mathcal{D}_{f,g}$  and let  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D}) = \sqcup_\lambda \Sigma_\lambda$  be the decomposition in connected components of the singular set  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D})$ .*

*Then for any symmetric homogeneous polynomial  $\varphi \in \mathbb{C}[z_1, \dots, z_{n-1}]$  of degree  $n-1$  there exist complex numbers  $\text{Res}_\varphi(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; \Sigma_\lambda)$  such that*

$$\sum_\lambda \text{Res}_\varphi(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; \Sigma_\lambda) = \int_S \varphi(TM - [S]^{\otimes \nu}).$$

We call Theorem 6.2 a *Lehmann-Suwa-type index theorem* since Lehmann and Suwa have introduced this kind of residues and the partial connection (4.2), which inspires the one in Proposition 6.1 (see [16], [17] and also [14]).

**Remark 6.3.** If we consider the couple  $(f, \text{Id}_M)$  then Theorem 6.2 turns out to be [3, Th.6.3.]. Observe that they assume also  $S'$  comfortably embedded into  $M$  but it is not necessary.

Like in Section 3 and 5, we conclude by deriving explicit formulas for the residues in Theorem 6.2 at isolated singularities. As usual we briefly recall how the residues are defined in Lehmann-Suwa theory (for a reference see [20, Sec.IV.5.]).

This kind of residues are defined very similarly to the ones in Section 3. Let  $\varphi \in \mathbb{C}[z_1, \dots, z_{n-1}]$  be any symmetric homogeneous polynomial of degree  $n-1$  and  $\delta_{\mathcal{D}}^{ls}$  the partial holomorphic connection on  $N_{\mathcal{D}}^M$  defined as in Proposition 6.1. Let  $\nabla_{\mathcal{D}}^{ls}$  be any  $(1,0)$ -type extension of it and  $\nabla_1^0$  and

$\nabla_2^0$  some connections respectively on  $N_{S^0}^{\otimes \nu}$  and  $TM|_{S^0}$  such that the triple  $(\nabla_1^0, \nabla_2^0, \nabla_{\mathcal{D}}^{ls})$  is compatible with the short exact sequence

$$0 \longrightarrow N_{S^0}^{\otimes \nu} \xrightarrow{\mathcal{D}|_{S^0}} TM|_{S^0} \xrightarrow{\rho} N_{\mathcal{D}}^M \longrightarrow 0. \quad (6.2)$$

Then set  $\nabla_{\mathcal{D}}^\bullet = (\nabla_1^0, \nabla_2^0)$  and observe that

$$\varphi(\nabla_{\mathcal{D}}^\bullet) = \varphi(\nabla_{\mathcal{D}}^{ls}) = 0$$

by the compatibility of the triple and the Bott vanishing theorem. This is the reason behind these choices. Let now  $\Sigma$  be a connected component of  $\text{Sing}(S) \cup \text{Sing}(\mathcal{D})$  and  $U \subset M$  an open subset such that  $U \cap (\text{Sing}(S) \cup \text{Sing}(\mathcal{D})) = \Sigma$ . Choose any connections  $\nabla_1^U$  and  $\nabla_2^U$  on  $[S]|_U^{\otimes \nu}$  and  $TM|_U$  and set  $\nabla_U^\bullet = (\nabla_1^U, \nabla_2^U)$ . Finally, let  $\tilde{R} \subset U$  be any compact real submanifold of dimension  $2n$  oriented as  $M$  and such that  $\Sigma \subset \text{int}(\tilde{R})$ . Assume its boundary  $\partial \tilde{R}$  transverse to  $S$  and with the orientation induced by  $\tilde{R}$ . Set  $R = \tilde{R} \cap S$  and  $\partial R = \partial \tilde{R} \cap S$ . Then by definition the residue is

$$\text{Res}(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; \Sigma) = \int_R \varphi(\nabla_U^\bullet) - \int_{\partial R} \varphi(\nabla_{\mathcal{D}}^\bullet, \nabla_U^\bullet),$$

where  $\varphi(\nabla_{\mathcal{D}}^\bullet, \nabla_U^\bullet)$  is the Bott difference form, defined as at [20, pp.71-72]. One can show that this formula does not depend on the choices done.

In particular if  $\Sigma = \{p\}$  is an isolated point we can shrink  $U$  so that  $[S]|_U^{\otimes \nu}$  and  $TM|_U$  are trivial and we can assume  $\nabla_1^U$  and  $\nabla_2^U$  trivial respect to some local frames. Then  $\varphi(\nabla_U^\bullet) = 0$  and the formula reduces to

$$\text{Res}(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; p) = - \int_{\partial R} \varphi(\nabla_{\mathcal{D}}^\bullet, \nabla_U^\bullet).$$

Assume now  $p \in \text{Sing}(\mathcal{D}) - \text{Sing}(S)$  and let  $(U, z) \in \mathfrak{U}$  be a local chart at  $p$ . Observe that  $N_{S^0}^{\otimes \nu}$  and  $TM|_{S^0}$  are not in general  $N_{S^0}^{\otimes \nu}$ -bundles but

$$N_{S^0}^{\otimes \nu}|_{U \cap S^0} \quad \text{and} \quad TM|_{U \cap S^0}$$

are canonically  $N_{U \cap S^0}^{\otimes \nu}$ -bundles, thanks to the natural ‘holomorphic actions’ of  $X$  on them given respectively by the Lie brackets

$$[X, \cdot] \quad \text{and} \quad [\mathcal{X}, \cdot]|_{U \cap S^0}.$$

These actions induce partial holomorphic connections on  $N_{S^0}^{\otimes \nu}|_{U \cap S^0}$  and  $TM|_{U \cap S^0}$  along  $N_{U \cap S^0}^{\otimes \nu}$  which are, together with  $\delta_{\mathcal{D}}^{ls}$ , compatible with (6.2) restricted to  $U \cap S^0$ . Therefore, similarly to Section 3, we can assume  $\nabla_1^0$  and  $\nabla_2^0$  to be ‘ $X$ -connections’ defined on  $U \cap S^0$  (and such that the triple  $(\nabla_1^0, \nabla_2^0, \nabla_{\mathcal{D}}^{ls})$  is compatible with (6.2)), and one can show that in this case

$$\text{Res}(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; p) = - \int_{\partial R} \varphi(\nabla_2^0, \nabla_2^U).$$

Thus we can work again as in the proof of [20, Th.III.5.5.] and obtain a formula like the one in [20, Th.IV.5.3.]. In particular, letting the  $h^j \in \mathcal{O}_{M,p}$

be as in (1.6) and taking  $\nabla_2^U$  trivial respect to the local frame  $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\}$  of  $TM$ , we get the formula

$$\text{Res}(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; p) = \int_{\Gamma} \frac{\varphi(-H)}{h^2 \dots h^n} dz^2 \wedge \dots \wedge dz^n, \quad (6.3)$$

where

$$H = \left( \frac{\partial h^j}{\partial z^k} \Big|_{U \cap S} \right)_{j,k=1,\dots,n}$$

and  $\Gamma$  is the  $(n-1)$  cycle defined as in Section 3. Observe that (6.3) is very similar to (3.3), there is just a little difference between the two matrices  $H$ .

**Remark 6.4.** Implicitly we have proved (and used) that

$$\text{Res}(\mathcal{D}; TM|_S - [S]|_S^{\otimes \nu}; p) = \text{Res}_{\varphi}(\mathcal{X}; TM|_S; p)$$

where the residue on the right is the one associated to the natural ‘holomorphic action’ of  $X$  on  $TM|_{U \cap S^0}$  mentioned just above (see also [20, Rmk.IV.5.7.(1)] or [11, Rmk.6.3.2(1)]). In particular, we could have used directly the formula of [20, Th.IV.5.3.].

Lastly, assume  $p \in \text{Sing}(S)$ . In this case we can work exactly as in the last part of Section 5. Assume to have an open neighborhood  $U \subset M$  of  $p$  such that  $U \cap (\text{Sing}(S) \cup \text{Sing}(\mathcal{D})) = \{p\}$  and  $g|_U$  is a biholomorphism onto its image. Suppose moreover to have a local generator  $y$  of  $\mathcal{I}_S$  and coordinates  $(u^1, \dots, u^n)$  on it, with  $(v^1 = u^1 \circ g^{-1}, \dots, v^n = u^n \circ g^{-1})$  the corresponding ‘special’ coordinates on  $g(U)$  at  $f(p)$ . Let  $\mathcal{V}_{f,g}$  be the holomorphic vector field on  $U$  defined as in (5.8), then reminding (6.1) and that  $\nu > 1$ , equation (5.9) implies also that the natural ‘holomorphic action’ of  $\mathcal{V}_{f,g}$  on  $N_{\mathcal{D}}^M|_{U \cap S^0}$  by Lie bracket induces locally  $\delta_{\mathcal{D}}^{ls}$ . Hence if we choose the coordinates  $(u^1, \dots, u^n)$  in such a way that  $\{y = (v^2 \circ f - v^2 \circ g)/y^{\nu} = \dots = (v^n \circ f - v^n \circ g)/y^{\nu} = 0\} = \{p\}$  then we can express the residue at  $p$  by the formula in [20, Th.IV.5.3.]. In particular, we get the formula

$$\text{Res}(\mathcal{D}; S; p) = \int_{\Gamma} \frac{y^{\nu(n-1)} \varphi(-Y)}{\prod_{j=2}^n (v^j \circ f - v^j \circ g)} du^2 \wedge \dots \wedge du^n, \quad (6.4)$$

where  $Y$  is the Jacobian matrix of

$$((v^2 \circ f - v^2 \circ g)/y^{\nu}, \dots, (v^n \circ f - v^n \circ g)/y^{\nu})$$

with respect to the coordinates  $(u^1, \dots, u^n)$  and  $\Gamma$  is the  $(n-1)$  cycle

$$\Gamma = \left\{ q \in U \cap S \text{ s.t. } \left| \frac{v^j \circ f - v^j \circ g}{y^{\nu}}(q) \right| = \epsilon, \text{ for } j = 2, \dots, n \right\},$$

for  $\epsilon > 0$  small enough, oriented as usual.

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